Lecture 42

(42.0) Notation : $\mathrm{Aut}(\Omega) = \{ f : \Omega \to \Omega \text{ holomorphic bijection} \}$
– conformal automorphisms of Ω.

Recall that we proved :

- $\mathrm{Aut}(\mathbb{D}) = \left\{ z \mapsto e^{i\phi} \frac{z-\alpha}{1-\bar{\alpha} z} \; : \; \alpha \in \mathbb{D}, \; \phi \in \mathbb{R} \pmod{2\pi} \right\}$

  (see Lecture 17 – § 17.3)

  $\mathbb{D} = D(0;1) = \{ z \in \mathbb{C} : |z| < 1 \}$ is the unit disc.

- $\mathrm{Aut}(\mathbb{C}) = \{ z \mapsto az+b \; : \; a, b \in \mathbb{C}, \; a \neq 0 \}$ (§ 41.0)

For an invertible 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we defined

$$M_A(z) = \frac{az+b}{cz+d}$$

(Möbius transformation)
[August Ferdinand Möbius 17/11/1790 – 26/09/1868]

Properties : $M_{A_1} \circ M_{A_2} = M_{A_1 A_2}$

$$M_A = Id \iff A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \quad (\lambda \in \mathbb{C} ; \lambda \neq 0)$$

(i.e. $M_A(z) = z \;\; \forall z \in \mathbb{C}$)

- $\mathrm{Aut}(\hat{\mathbb{C}}) = \{ M_A \; : \; A \in \underbrace{GL_2(\mathbb{C})}_{\text{"group" of } 2\times 2 \text{ invertible matrices}} \}$

(here $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere)

- Möbius transformations preserve circles and lines in the complex plane – or, equivalently, circles on the Riemann sphere $\hat{\mathbb{C}}$ (lines in $\mathbb{C}$ = circles in $\hat{\mathbb{C}}$ passing through ∞)

(42.1) Fixed points of Möbius transformations.

Lemma. Let M be a Möbius transformation, $M \neq Id$.

Then the equation $M(z) = z$ has at most 2 solutions.

Proof. $$M(z) = \frac{az+b}{cz+d} = z \Rightarrow az+b = cz^2+dz.$$

i.e., $$cz^2 + (d-a)z - b = 0$$

If $c \neq 0$, this equation has one or two solutions, depending on $(d-a)^2 - 4bc = 0$ or $\neq 0$.

If $c = 0$, scaling d to be 1, $M(z) = az+b$ has exactly one fixed point $M(\infty) = \infty$ - if $b \neq 0$
two fixed points $M(0) = 0$; $M(\infty) = \infty$ if $b = 0$. □

(42.2) Theorem. – Given three distinct points z_1, z_2, z_3 on $\hat{\mathbb{C}}$, there exists a unique Möbius transformation M such that $M(z_1) = 0$, $M(z_2) = 1$, $M(z_3) = \infty$.

In other words, any triple (z_1, z_2, z_3) can be sent to $(0, 1, \infty)$ via Möbius transformations – hence any two triples (z_1, z_2, z_3) ; (w_1, w_2, w_3) can be mapped to one another via Möbius transformations.

Proof: We have to consider a few cases – depending on whether $\infty \in \{z_1, z_2, z_3\}$ or not.

Case 1. $\infty \notin \{z_1, z_2, z_3\}$. Take $M(z) = \dfrac{z_2 - z_3}{z_2 - z_1} \cdot \dfrac{z - z_1}{z - z_3}$.

Case 2. $z_1 = \infty$. Take $M(z) = \dfrac{z_2 - z_3}{z - z_3}$.

Case 3. $z_2 = \infty$. Take $M(z) = \dfrac{z - z_1}{z - z_3}$.

Case 4. $z_3 = \infty$. Take $M(z) = \dfrac{z - z_1}{z_2 - z_1}$.

Uniqueness. If M_1 and M_2 are two Möbius transformations sending (z_1, z_2, z_3) to $(0, 1, \infty)$, then $M_1^{-1} M_2 = M$ has at least 3 fixed points $(M(z_j) = z_j \ ; \ j = 1, 2, 3)$. Hence, by Lemma 42.1, $M = Id$, i.e. $M_1 = M_2$.

$\square$

(42.3) Corollary to Theorem (42.2). I.

Let K_1 and K_2 be two circles in the complex plane. Then there exists a Möbius transformation mapping K_1 to K_2.

Proof. - Take 3 points z_1, z_2, z_3 on K_1
w_1, w_2, w_3 on K_2.

Let M be ~~a~~ the Möbius transformation s.t. $M(z_j) = w_j$ $(j = 1, 2, 3)$.

Then $M(K_1)$ is a circle passing through w_1, w_2, w_3.
Since three points (non-collinear) determine a circle, $M(K_1) = K_2$. $\square$

(42.4) Corollary to Theorem (42.2) II. - Cross Ratio.

Let z_1, z_2, z_3, z_4 be 4 distinct points on $\hat{\mathbb{C}}$. We define $[z_1 : z_2 : z_3 : z_4]$ - called <u>the cross ratio</u> - to be the complex number $M(z_4)$ where M is the unique Möbius transformation sending (z_1, z_2, z_3) to $(0, 1, \infty)$.

In more detail: - see the proof of Theorem (42.2) -

- if $\infty \notin \{z_1, z_2, z_3\}$: $[z_1 : z_2 : z_3 : z_4] = \dfrac{z_2 - z_3}{z_2 - z_1} \dfrac{z_4 - z_1}{z_4 - z_3}$

$$= - \frac{(z_1 - z_4)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z_4)}$$

(if $z_4 = \infty$, we get $\dfrac{z_2 - z_3}{z_2 - z_1}$)

- $$[\infty : z_2 : z_3 : z_4] = \frac{z_2 - z_3}{z_4 - z_3}.$$

- $$[z_1 : \infty : z_3 : z_4] = \frac{z_4 - z_1}{z_4 - z_3}.$$

- $$[z_1 : z_2 : \infty : z_4] = \frac{z_4 - z_1}{z_2 - z_1}.$$

(Cross-ratio is an invariant of Möbius transformations – see 42.7 below)

(42.5) Converse to Corollary (42.3). Let $\Omega \subset \mathbb{C}$ be an open connected set; $f : \Omega \to \mathbb{C}$ a holomorphic function.

Assume that there is a circle $C(\alpha; r)$ in Ω s.t.

(1) Ω contains the open disc $D(\alpha; r)$

(2) f maps $C(\alpha; r)$ to a circle in $\mathbb{C}$ bijectively.

Then f is a Möbius transformation.

Proof. Let $K = C(\alpha; r)$ and $K' = f(K)$. By Lemma (40.3) f is an equivalence (conformal) between the discs bounded by K and K'. Let M_1, M_2 be Möbius transformations s.t.

$$M_1 : C(0;1) \mapsto K \quad ; \quad M_2 : K' \mapsto C(0;1)$$

Then
$$M_2 \circ f \circ M_1 \in \mathrm{Aut}(\mathbb{D}).$$

Now every element of $\mathrm{Aut}(\mathbb{D})$ is a Möbius transformation

$$\Rightarrow M_2 \circ f \circ M_1 = M \Rightarrow f = M_2^{-1} \circ M \circ M_1^{-1} \text{ is}$$

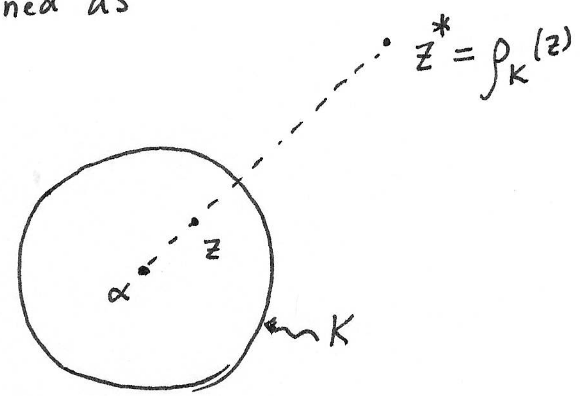
a Möbius transformation $\square$

(42.6) Relation with reflection relative to a circle / line.

Let $\alpha \in \mathbb{C}$ and $r \in \mathbb{R}_{>0}$; and $K = C(\alpha; r)$.

Definition.– $\rho_K : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is defined as

$$\rho_K(z) = \alpha + \frac{r^2 (z-\alpha)}{|z-\alpha|^2} \; ; \; z \neq \alpha, \infty$$



$$\rho_K(\alpha) = \infty \; ; \; \rho_K(\infty) = \alpha.$$

Note: (1) $z^* = \rho_K(z)$ is the unique point on the ray $\alpha + \mathbb{R}_{>0}(z-\alpha)$ s.t. $|z-\alpha| \cdot |z^* - \alpha| = r^2$. To see this, we put $z^* - \alpha = \lambda(z-\alpha)$ $(\lambda \in \mathbb{R}_{>0})$ in $|z-\alpha|\,|z^*-\alpha| = r^2$ to get $\lambda = \dfrac{r^2}{|z-\alpha|^2}$ $\Rightarrow$ $z^* = \alpha + \dfrac{r^2 (z-\alpha)}{|z-\alpha|^2}$

(2) $\rho_K(z) = \alpha + \dfrac{r^2}{\bar{z} - \bar{\alpha}}$ is NOT holomorphic – but "anti-holomorphic" – i.e. it reverses the orientation. (while preserving angles)

(3) $\rho_K(\rho_K(z)) = z \quad \forall z \in \hat{\mathbb{C}}$.

i.e. $\rho_K^{-1} = \rho_K$.

Theorem. – Let K be a circle in the complex plane, M a Möbius transformation; $K' = M(K)$. Then

$$M \circ \rho_K = \rho_{K'} \circ M$$

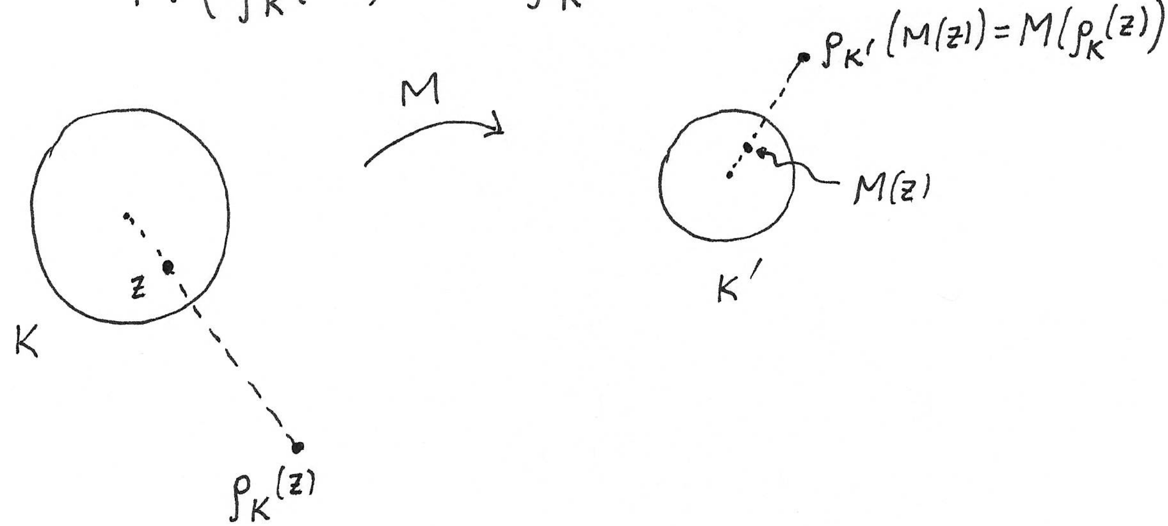
Proof. – Let $f = M^{-1} \circ \rho_{K'} \circ M \circ \rho_K$. Then f is holomorphic (even number of orientation reversal $\Rightarrow$ orientation preservation);

maps $K \mapsto K$, hence a Möbius transformation.

But $f(z) = z \;\; \forall z \in K \;\; \Rightarrow$ (by Lemma 42.1) $f = Id$. $\square$

In other words, for every $z \in \mathbb{C}$; $z \neq$ center of K,

$$M(\rho_K(z)) = \rho_{K'}(M(z))$$



(42.7) Let (z_1, z_2, z_3, z_4) and (w_1, w_2, w_3, w_4) be two quadruples of points on $\hat{\mathbb{C}}$. Then $\exists$ a Möbius transformation M such that $M(z_j) = w_j \quad (j = 1,2,3,4)$

$$\Longleftrightarrow \quad [z_1 : z_2 : z_3 : z_4] = [w_1 : w_2 : w_3 : w_4]$$

Proof. Let M_1, M_2 be the unique Möbius transformations

s.t. $\quad M_1 : (z_1, z_2, z_3) \longmapsto (0, 1, \infty)$

$\quad\quad M_2 : (w_1, w_2, w_3) \longmapsto (0, 1, \infty)$

If $M(z_j) = w_j \quad (j = 1, 2, 3)$ then $M = M_2^{-1} \circ M_1$ by uniqueness.

Hence
$$M(z_4) = w_4$$
$$\Longleftrightarrow \quad M_2^{-1}(M_1(z_4)) = w_4$$
$$\Longleftrightarrow \quad M_1(z_4) = M_2(w_4).$$

Now we are done, since $M_1(z_4) = [z_1 : z_2 : z_3 : z_4]$

$M_2(w_4) = [w_1 : w_2 : w_3 : w_4]$ $\square$