Lecture 43

(43.0) Recall that we proved (§41.5): $\mathrm{Aut}(\hat{\mathbb{C}}) =$ Möbius transformations:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightsquigarrow M_A(z) = \frac{az+b}{cz+d}$$ (see Lecture 42 for main properties of Möbius transformations)

$ad - bc \neq 0$

Remarks. (1) Let K be a circle in the complex plane. Then:

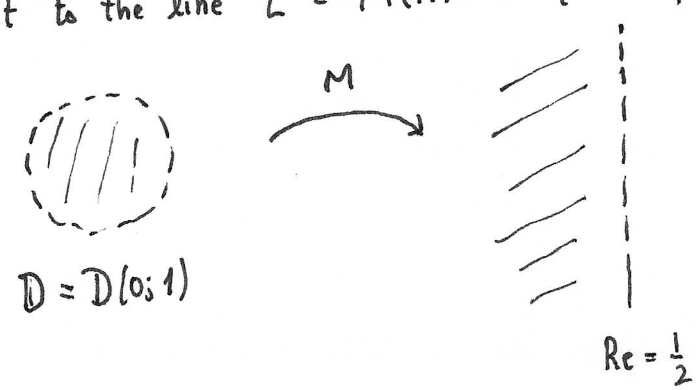
- if $c \neq 0$ and $-\frac{d}{c} \in K$, then $M_A(K)$ is a straight line
- $-\frac{d}{c} \notin \mathrm{Interior}(K) \Rightarrow M_A : \mathrm{Interior}(K) \to \mathrm{Interior}(M_A(K))$ (bijective, holomorphic i.e. conformal equivalence)

e.g. let $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ i.e. $M(z) = \dfrac{z}{z+1}$. $K = C(z_0, R)$.

i.e. $K = \{ z \in \mathbb{C} : |z - z_0| = R \}$ $(z_0 \in \mathbb{C} ; R \in \mathbb{R}_{>0})$.

- if -1 lies on K - say e.g. $K = C(0;1)$, $M(K)$ is a straight line: $M(-1) = \infty$, $M(1) = \frac{1}{2}$, $M(i) = \dfrac{i}{1+i} = \dfrac{i(1-i)}{(1+i)(1-i)} = \frac{1}{2}(1+i)$

As $M(0) = 0$, the interior of K is mapped bijectively to the left half plane (left to the line $L = M(K) = \{ \text{Real part} = \frac{1}{2} \}$).



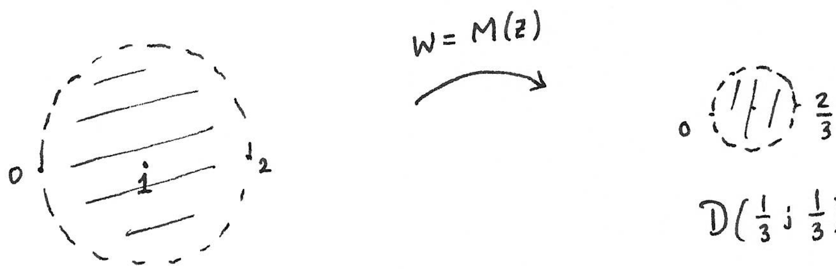
- $M(z) = \dfrac{z}{z+1} = 1 - \dfrac{1}{z+1} = \left(\tau_1 \circ \sigma_{-1} \circ I \circ \tau_1\right)(z).$

Using the formulae listed in § 41.3, one can easily check that

M maps $C(z_0 ; R)$ to $C\left(1 + \dfrac{\bar{z}_0 + 1}{R^2 - |z_0+1|^2} ; \dfrac{R}{|R^2 - |z_0+1|^2|}\right)$ if $|z_0+1| \neq R$.

For instance, let $K = C(1;1)$ (so $-1 \notin$ Interior (K)).

$$M(K) = C\left(\tfrac{1}{3} ; \tfrac{1}{3}\right).$$



$D(\tfrac{1}{3} ; \tfrac{1}{3}) = \{w \mid |w - \tfrac{1}{3}| = \tfrac{1}{3}\}$

$D(1;1) = \{z \mid |z-1| < 1\}$

(2) $M(z) = \dfrac{az+b}{cz+d}$ implies $\dfrac{d}{dz} M(z) = \dfrac{ad-bc}{(cz+d)^2} \neq 0$.

Hence M preserves angles everywhere.

Note: We may assume that $ad-bc = 1$, since $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = A$ and

$B = \begin{bmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{bmatrix}$ define the same Möbius transformation and

$$\lambda^2 \det(A) = \det(B).$$

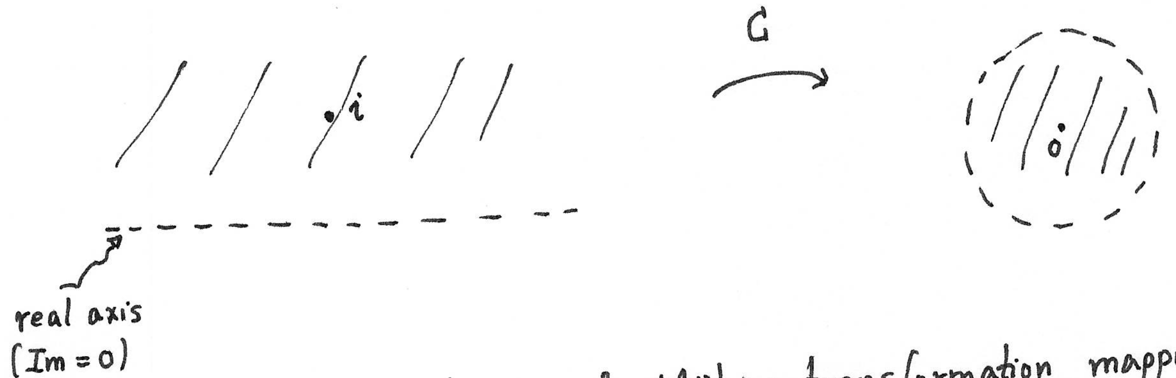
If $\Delta = ad - bc \neq 0$, take $\lambda = (\sqrt{\Delta})^{-1}$ to get $\det(B) = 1$.

In this case, $|M'(z)| = \dfrac{1}{|cz+d|^2}$, hence M <u>stretches the part</u> $\{z \mid |cz+d| < 1\}$ and <u>shrinks the part</u> $\{z \mid |cz+d| > 1\}$.

(43.1) Glossary of some conformal equivalences.

- Cayley transform : $C(z) = \dfrac{z-i}{z+i} : \mathbb{H} \xrightarrow{\sim} \mathbb{D}$.

$\mathbb{H} = \{ z \mid \mathrm{Im}(z) > 0 \}$; $\mathbb{D} = \{ z \mid |z| < 1 \}$.



$C(i) = 0$.

Note : The most general Möbius transformation mapping $\mathbb{H}$ bijectively to $\mathbb{D}$; say M can be written as follows.

Let $\alpha \in \mathbb{H}$ be an arbitrary point. Assuming $M(\alpha) = 0$ - we conclude that reflection of α in the boundary of $\mathbb{H}$, $\partial\mathbb{H} = \mathbb{R}$ - must be mapped to reflection of 0 in the circle $C(0;1)$.

i.e. $M(\bar{\alpha}) = \infty$. Hence $M = \lambda \cdot \dfrac{z-\alpha}{z-\bar{\alpha}}$.

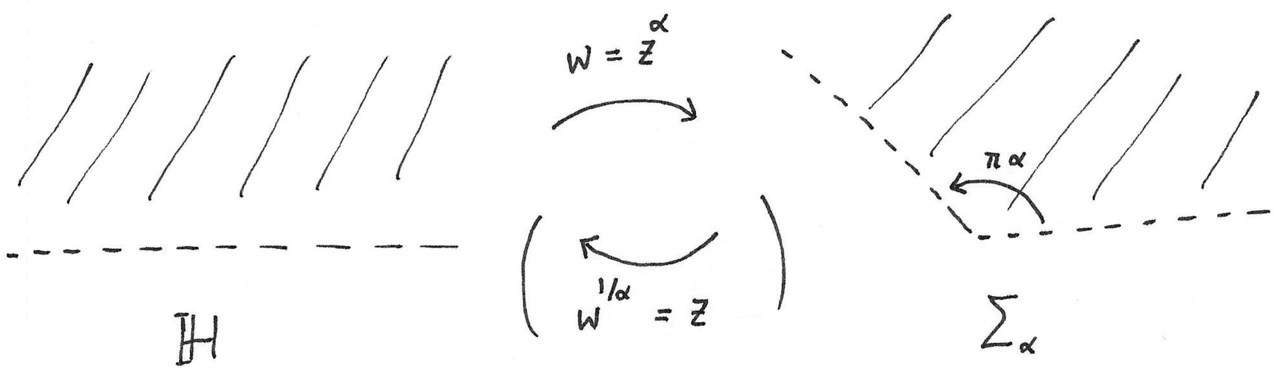
Since $M : \partial\mathbb{H} \to \partial\mathbb{D}$, we conclude that $|\lambda| = 1$.

Conformal equivalences $\mathbb{H} \to \mathbb{D}$ $= \left\{ z \mapsto \lambda \dfrac{z-\alpha}{z-\bar{\alpha}} : \begin{array}{l} \alpha \in \mathbb{H} \\ |\lambda| = 1 \end{array} \right\}$.

- Power function : $z \mapsto z^{\alpha}$ $(0 < \alpha < 1)$ one

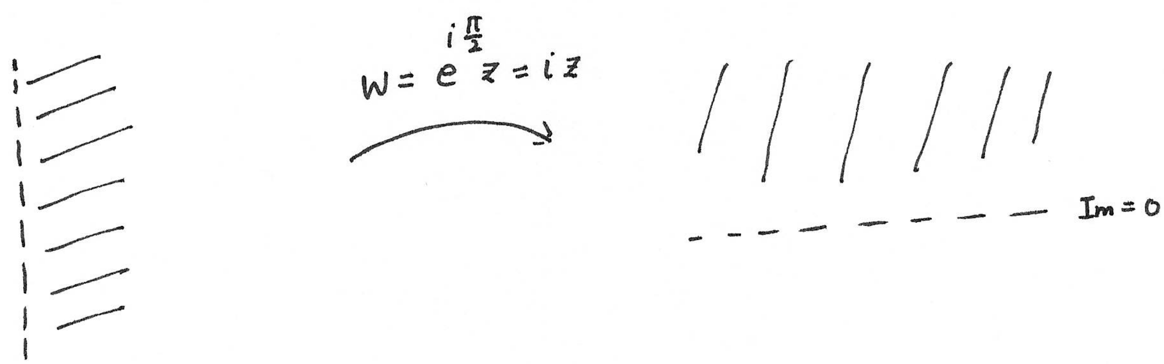
Recall that $z^{\alpha} = \exp(\alpha \cdot \log(z))$ is defined on $\mathbb{C} \setminus \mathbb{R}_{\leq 1}$ - by taking the standard determination of log.

Thus $z = r \cdot e^{i\theta} \rightsquigarrow z^\alpha = r^\alpha e^{i\theta\alpha}$ sets up a conformal equivalence between $\mathbb{H}$ and $\Sigma_\alpha = \{ z \mid 0 < \arg(z) < \pi\alpha \}$



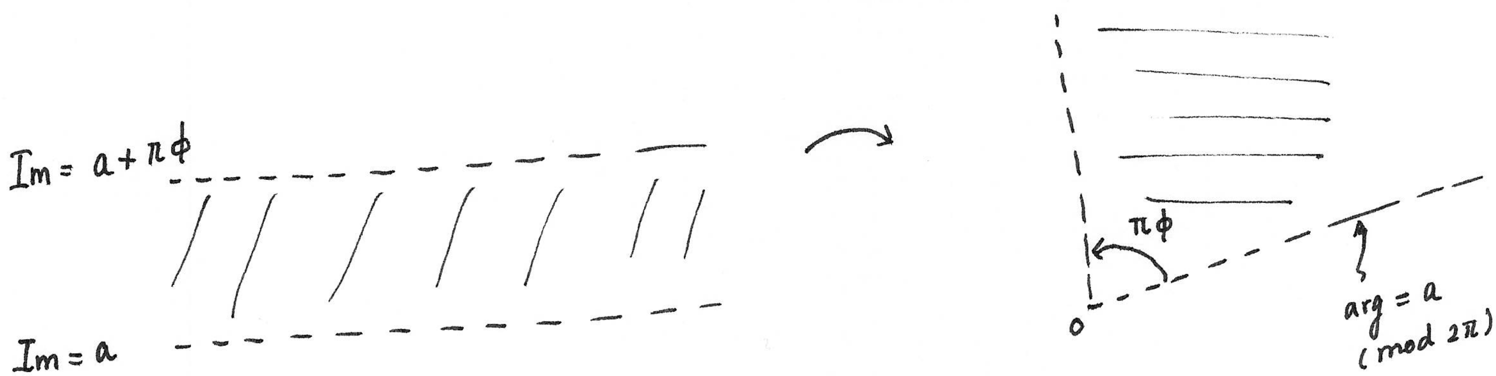
- Rotation by ϕ : $z \longmapsto e^{i\phi} \cdot z$ rotates a given domain - counterclockwise by ϕ.

e.g



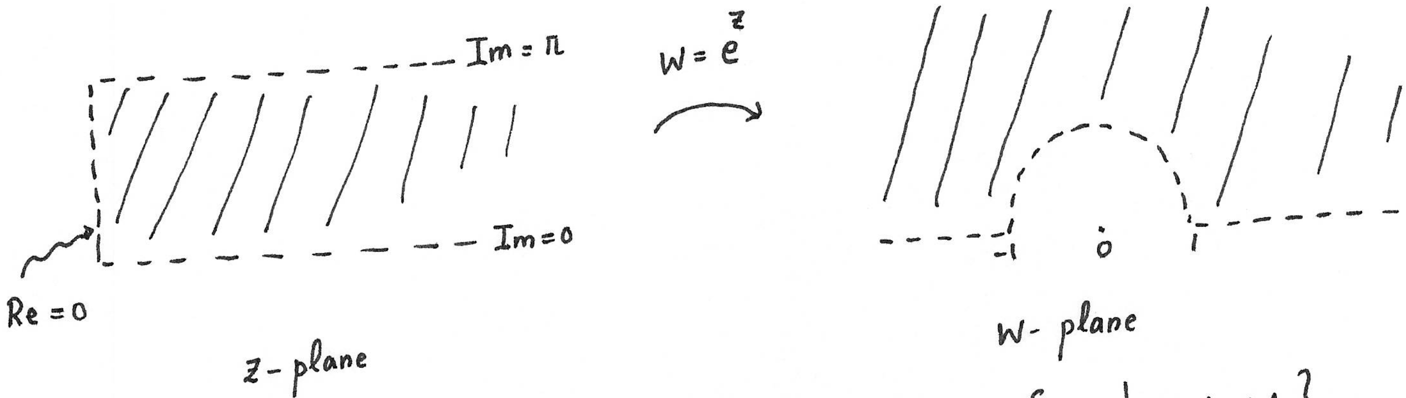
Re > 0

- Exponential function. $w = e^z$ - maps a horizontal strip of width < 2π to a sector.



$(0 < \phi < 2)$

e.g. let $\Omega = \{ z \mid \mathrm{Re}(z) > 0 \,;\ 0 < \mathrm{Im}(z) < \pi \}$.

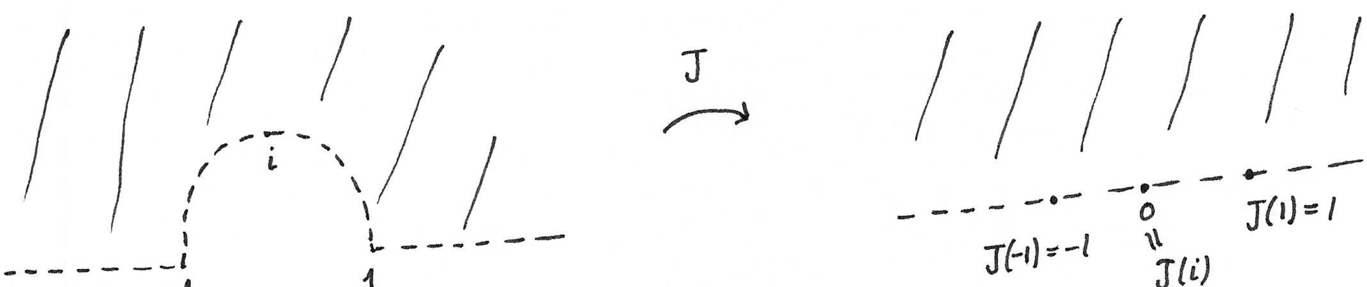


$\mathbb{H} \setminus \{ w \mid |w| \leq 1 \}$

(Joukowski domain)

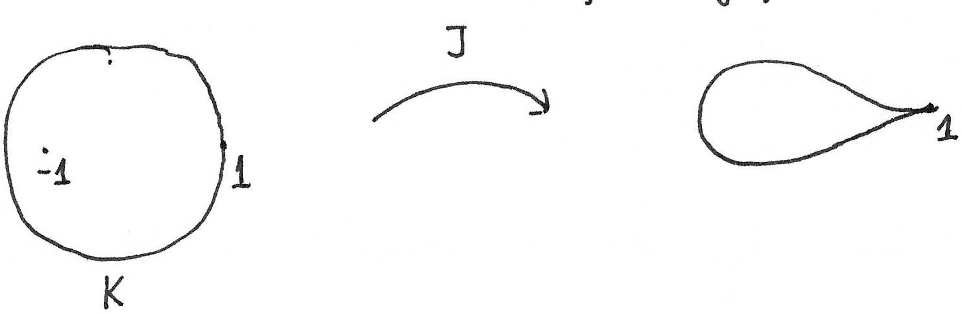
also spelled as Zhoukovskii.

- Joukowski transformation. $J(z) = \dfrac{z + z^{-1}}{2}$ (sometimes people do not scale it and use $z + z^{-1}$)

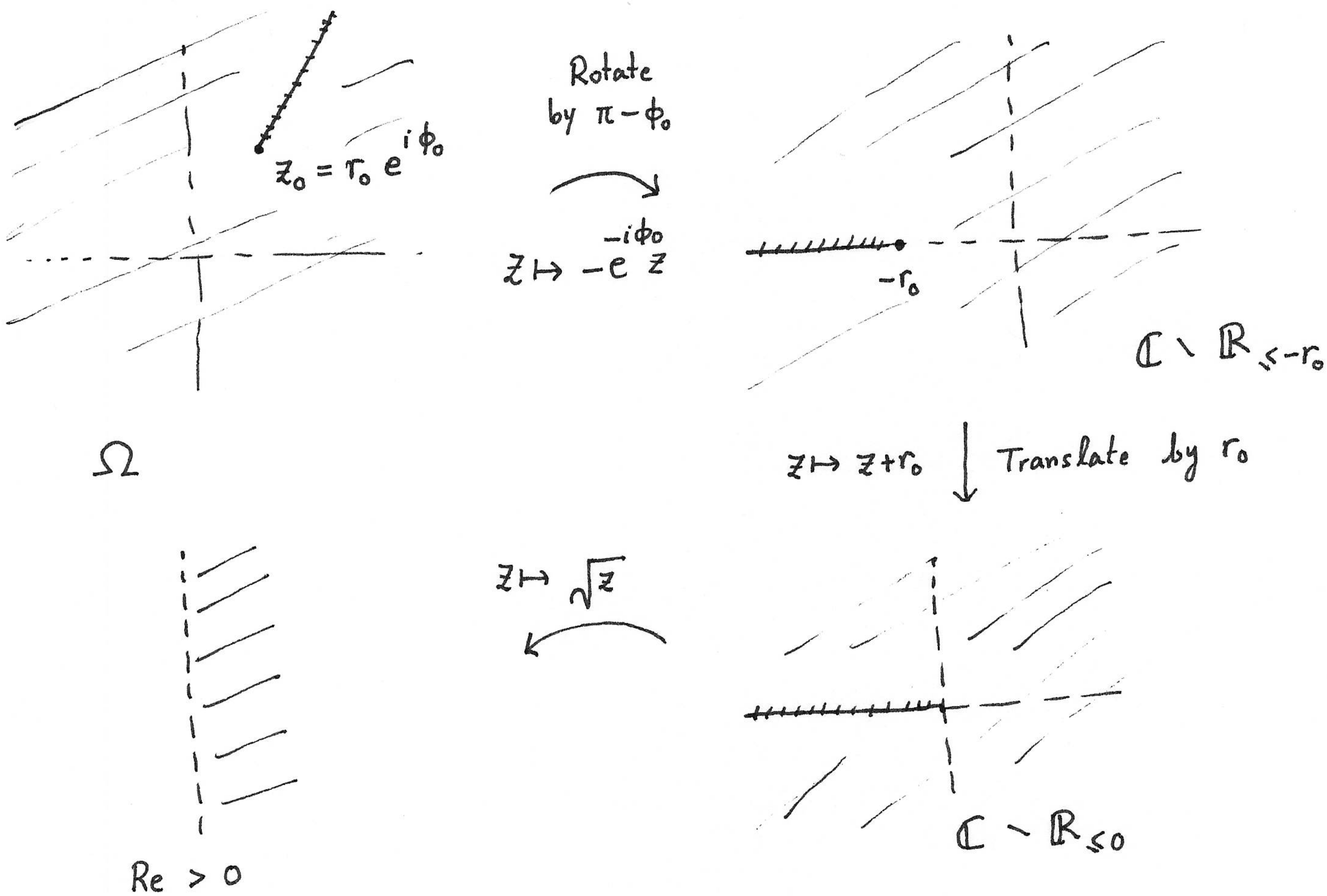


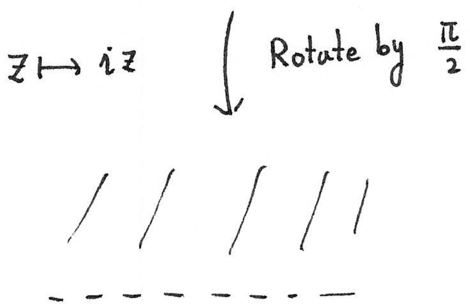
Remark. - $J(z)$ transforms $C(0;1)$ - unit circle to $[-1, 1]$.

If K is any other circle passing through 1 - $J(K)$ turns into a curve (called airwing profile) - see e.g. the link I added on Carmen. for its significance to aerospace engineering.



(43.2) Examples. - I. Cut plane. Let $z_0 \in \mathbb{C}$ and consider the domain $\Omega = \mathbb{C} \setminus \mathbb{R}_{\geq 1} \cdot z_0$.



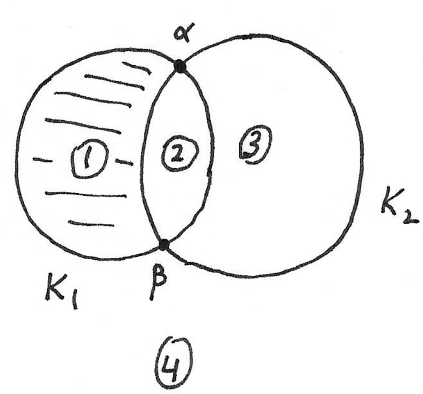


Composing all these, we get

$$f(z) = i\sqrt{r_0 - e^{-i\phi_0} z} \quad : \Omega \xrightarrow{\sim} \mathbb{H}$$

II. Circular lune shapes.

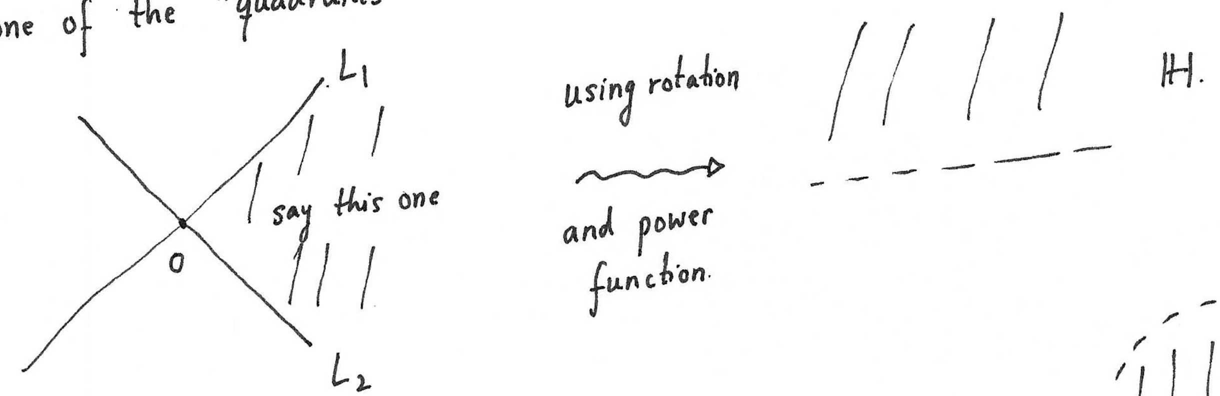
Let K_1 and K_2 be two circles intersecting at α and β.



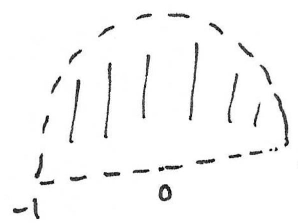
Ω = region shaded (marked 1) in this picture

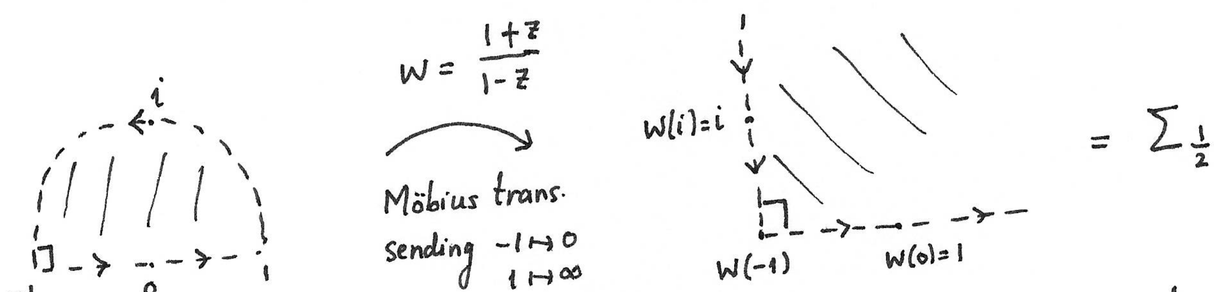
Let M be a Möbius transformation s.t. $M(\alpha) = \infty$
$M(\beta) = 0$.

Then $L_j = M(K_j)$ are straight lines meeting at 0. M transforms Ω to one of the "quadrants"



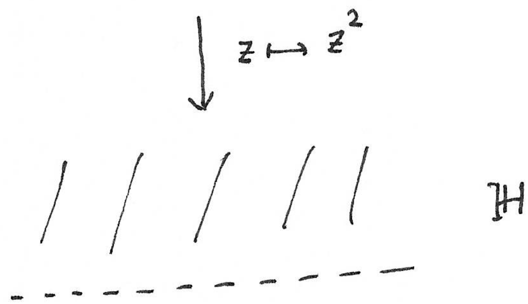
e.g. $\Omega = \{ z \mid |z| < 1 \text{ and } \operatorname{Im}(z) > 0 \}$





$$w(i) = \frac{1+i}{1-i} = \frac{(1+i)^2}{2} = i$$

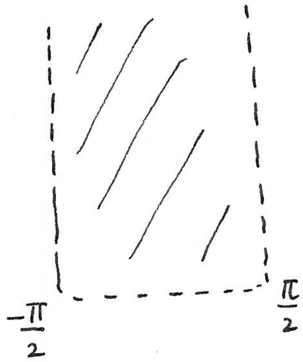
(Note: angles are preserved at -1)



Hence $f(z) = \left(\frac{1+z}{1-z} \right)^2$

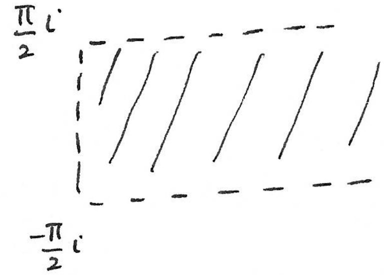
is a conformal equivalence $\Omega \xrightarrow{\sim} \mathbb{H}$.

III $\sin(z)$. Consider $\Omega = \left\{ z \;\middle|\; -\frac{\pi}{2} < Re(z) < \frac{\pi}{2} \; ; \; Im(z) > 0 \right\}$.

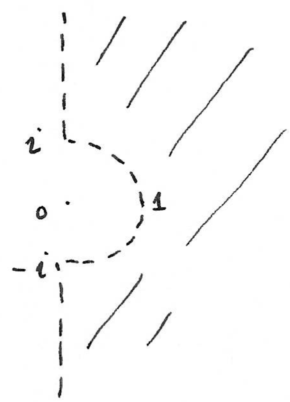


$z \mapsto -iz$

rotate clockwise by $\frac{\pi}{2}$

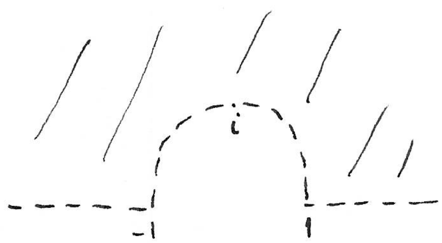


$z \mapsto e^{z}$



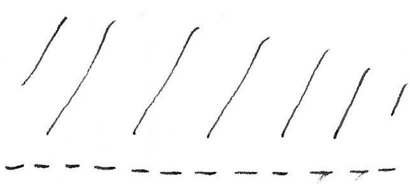
Rotate counterclockwise by $\frac{\pi}{2}$

$z \mapsto iz$



Joukowski domain

$z \mapsto J(z) = \frac{z+z^{-1}}{2}$



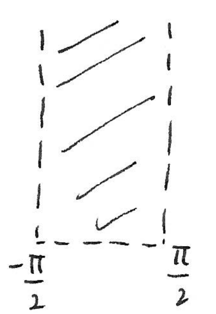
$\mathbb{H}$

Composing these we get

$$f(z) = \frac{1}{2}\left(i\, e^{-iz} + \frac{1}{i\, e^{-iz}} \right)$$

$$= \frac{1}{2i}\left(e^{iz} - e^{-iz} \right) = \sin(z)$$

$\Rightarrow$ $\sin(z)$:



Conformal equivalence