Lecture 44

(44.0) Riemann mapping theorem. Let $\Omega \subsetneq \mathbb{C}$ be a proper, open, connected and simply-connected set. Then there exists a conformal equivalence $f : \Omega \to \mathbb{D}$ (i.e. f is a holomorphic bijection).

$\mathbb{D}$ = unit disc $= \{z \mid |z|<1\}$

Uniqueness of f can be obtained as follows: pick $z_0 \in \Omega$. Then f is unique if we require $f(z_0)=0$ and $f'(z_0) \in \mathbb{R}_{>0}$.

Proof of uniqueness: Assume f_1 and f_2 are two conformal equivalences

$$f_j : \Omega \to \mathbb{D} \quad ; \quad f_j(z_0) = 0 \quad \text{and} \quad f_j'(z_0) \in \mathbb{R}_{>0}.$$

Then $g : \mathbb{D} \to \mathbb{D}$ given by $g(z) = f_2(f_1^{-1}(z))$ is a conformal automorphism of $\mathbb{D}$ and $g(0)=0$.

Recall that $\mathrm{Aut}(\mathbb{D}) = \left\{ z \mapsto e^{i\theta} \dfrac{z-\alpha}{1-\bar{\alpha}z} \ : \ \alpha \in \mathbb{D},\ \theta \in \mathbb{R} \bmod 2\pi \right\}$

$\Rightarrow$ $\exists\ \theta \in \mathbb{R}$ such that $g(z) = e^{i\theta} z$. That is, $g'(0) = e^{i\theta}$.

Since $g = f_2 \circ f_1^{-1}$ we get $g'(z) = f_2'(f_1^{-1}(z)) \cdot \dfrac{1}{f_1'(f_1^{-1}(z))}$ (by chain rule)

$$\Rightarrow \quad e^{i\theta} = g'(0) = \frac{f_2'(z_0)}{f_1'(z_0)} \in \mathbb{R}_{>0} \quad \Rightarrow \quad \theta \in 2\pi\mathbb{Z}, \text{ i.e. } g(z) = z \ \forall z \in \mathbb{D}$$

and hence $f_2 = f_1$. $\square$

(44.1) The existence part of Theorem 44.0 is a bit long and is based on the following list of results - which are of interest in their own right.

1. Schwarz' Lemma : Let $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic function
[Lecture 17 : § 17.1]
such that $f(0) = 0$ and $|f(z)| < 1$ $\forall\, z \in \mathbb{D}$. Then $|f(z)| \leqslant |z|$ for every $z \in \mathbb{D}$. Moreover if $\exists\, z_0 \in \mathbb{D}$, $z_0 \neq 0$ s.t. $|f(z_0)| = |z_0|$ then f is a rotation – i.e. $f(z) = \lambda \cdot z$ $(|\lambda| = 1)$.

2. If $\Omega \subset \mathbb{C}$ is an open, connected and simply-connected set, and $f : \Omega \to \mathbb{C}$ is a holomorphic, no where vanishing function, then
(i.e. $f(z) \neq 0$ $\forall\, z \in \Omega$)
there exists a holomorphic $g : \Omega \to \mathbb{C}$ s.t. $f(z) = e^{g(z)}$. In particular, taking $f_1(z) = e^{\frac{1}{2} g(z)}$ we get a holomorphic square-root of f – i.e. $f(z) = f_1(z)^2$ $\forall\, z \in \Omega$. [see Lecture 29; § 29.4]

3. If $f : \underset{\text{open \& connected}}{\Omega} \to \mathbb{C}$ is a 1-1, holomorphic function, then f is a conformal equivalence between Ω and $f(\Omega)$. [Open mapping and inverse function Thms. – Lecture 24 24.1, 24.3]

4. (Weierstrass' Theorem) – Let $\{f_n : \underset{\text{open \& connected}}{\Omega} \to \mathbb{C}\}_{n \geq 1}$ be a uniformly
[Lecture 19]
convergent sequence of functions. Then $f = \lim_{n \to \infty} f_n : \Omega \to \mathbb{C}$ is holomorphic.

Hurwitz' Theorem* : If each f_n is univalent (i.e. 1-1) then either f is constant or univalent.

(We haven't proved Hurwitz' Theorem yet.)

And finally, the following fundamental result from complex function theory (yet to be proved in this class)

* Adolf Hurwitz (26/3/1859 – 18/11/1919)

5. Montel's Theorem*. - ($\Omega \subset \mathbb{C}$ open, connected).

Let $\mathcal{F}$ be a set of holomorphic functions $\Omega \to \mathbb{C}$ such that :

$$\left[\begin{array}{l} \text{for every compact set } K \subset \Omega \text{, there exists a constant } M > 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(depending on } K\text{)} \\ \text{s.t.} \quad |f(z)| < M, \ \forall z \in K, \ f \in \mathcal{F} \end{array}\right] \quad - (*)$$

$$\left[\begin{array}{l} \text{Then, every sequence of functions } (f_n)_{n \geq 1} \subset \mathcal{F} \text{ has a uniformly} \\ \text{convergent subsequence - i.e. } \exists\ 1 \leq n_1 < n_2 < \dots \text{ s.t. } (f_{n_k})_{k \geq 1} \end{array}\right] \quad - (**)$$

converges uniformly.

Remark - Montel's Theorem is actually an <u>if and only if</u> statement, and is usually stated as: locally bounded $\Leftrightarrow$ normal :

- $\mathcal{F} \subset \{f : \Omega \to \mathbb{C} \text{ holomorphic}\}$ is <u>locally bounded</u> if $(*)$ above holds. $\mathcal{F}$ is <u>normal</u> if $(**)$ holds.

(44.2) Koebe's "stretching" function. (Paul Koebe 1882-1945)

<u>Theorem.</u> Assume that $\Omega_0 \subsetneq \mathbb{D}$ (proper, open, connected) ; $0 \in \Omega_0$

and : $\boxed{\begin{array}{l} \text{for every holomorphic, nowhere vanishing } h : \Omega_0 \to \mathbb{C} \setminus \{0\}, \\ \text{there exists a holomorphic square-root of } h \text{ - i.e. } h_1 : \Omega_0 \to \mathbb{C} \\ \text{s.t. } h_1(z)^2 = h(z) \ \forall z \in \Omega_0. \end{array}}$

Then $\exists$ a holomorphic, univalent $\mathcal{K} : \Omega_0 \to \mathbb{D}$ such that

$$\mathcal{K}(0) = 0 \quad \text{and} \quad |\mathcal{K}(z)| > |z| \ \ \forall z \in \Omega_0, z \neq 0.$$

*Paul Antoine Aristide Montel (29/4/1876 - 22/1/1975)

Proof. - Let $a \in \mathbb{D} \setminus \Omega_0$; $\boxed{\varphi_a(z) = \dfrac{z-a}{1-\bar{a}z} \in Aut(\mathbb{D})}$

Since $\varphi_a(z) \neq 0 \ \forall z \in \Omega_0$, and φ_a is univalent, we have a holomorphic square-root of φ_a. (by the hypothesis of the theorem).

$g : \Omega_0 \to \mathbb{C}$ s.t. $g(z)^2 = \varphi_a(z) \ \forall z \in \Omega_0$.

Note : g is univalent ($g(z_1) = g(z_2) \Rightarrow g(z_1)^2 = g(z_2)^2$ i.e. $\varphi_a(z_1) = \varphi_a(z_2)$
$\Rightarrow z_1 = z_2$ since φ_a is univalent)

and $|g(z)|^2 = |\varphi_a(z)| < 1$; i.e. $g : \Omega_0 \to \mathbb{D}$ is univalent.

Take $b = g(0)$ and define $\boxed{\kappa(z) = \varphi_b(g(z))}$. Clearly κ is univalent and $\kappa(0) = \varphi_b(g(0)) = \varphi_b(b) = \dfrac{b-b}{1-\bar{b}b} = 0$.

It remains to prove that $|\kappa(z)| > |z| \ \forall z \in \Omega_0 \setminus \{0\}$.

Let $\Omega_1 = \kappa(\Omega_0)$ so that $\kappa : \Omega_0 \xrightarrow{\sim} \Omega_1$. Let $h : \Omega_1 \to \Omega_0$ be its inverse - i.e. $\kappa = \varphi_b \circ g$ - so, $h = \kappa^{-1} = g^{-1} \circ \varphi_b^{-1} = g^{-1} \circ \varphi_{-b}$.

Note : $g^{-1}(z) = \varphi_{-a}(z^2) = \dfrac{z^2+a}{1+\bar{a}z^2}$ (inverse of $\sqrt{\ } = (\ )^2$)

hence h is defined on $\mathbb{D}$ and is not a rotation. So, by Schwarz' Lemma $|h(z)| < |z| \ \forall z \in \mathbb{D} \setminus \{0\}$.

Take $z = \kappa(w)$ ($w \in \Omega_0 \setminus \{0\}$) to get $|w| < |\kappa(w)|$

□

(44.3) Proof of Riemann Mapping Theorem (Koebe)

A slight relaxation of the hypotheses: let $\Omega \subsetneq \mathbb{C}$ be an open, connected set such that (see statement of Koebe's Theorem (44.2)).

$$\left[\begin{array}{l}\text{for every holomorphic, no-where vanishing } h : \Omega \to \mathbb{C}, \\ \text{there exists a holomorphic square-root - i.e. } h_1 : \Omega \to \mathbb{C} \\ \text{s.t. } h_1(z)^2 = h(z)\end{array}\right]$$

(Remark - simply-connectedness $\Rightarrow$ this assumption - see page 2, item a above).

<u>Step 1.</u> $\exists\ \beta : \Omega \xrightarrow{\simeq} \Omega_0 \subset \mathbb{D}\ ;\ 0 \in \Omega_0$.

Proof. - Let $a \in \mathbb{C} \setminus \Omega$ and let $g : \Omega \to \mathbb{C}$ be a square-root of $z - a$. $(g(z)^2 = z - a)$ - exists by our assumption and $a \notin \Omega$.

Note: • g is univalent, since $g(z_1) = g(z_2) \Rightarrow g(z_1)^2 = g(z_2)^2 \Rightarrow z_1 = z_2$.

• $w_0 \in g(\Omega) \Rightarrow -w_0 \notin g(\Omega)$

(if $w_0 = g(z_1)$ and $-w_0 = g(z_2)$, then again $g(z_1)^2 = g(z_2)^2$ $\Rightarrow z_1 = z_2$ and hence $w_0 = -w_0$ i.e. $0 \in g(\Omega)$ i.e. $a \in \Omega$ - contradiction)

Pick $w_0 \in g(\Omega)$. By Open mapping theorem, $\exists\ r > 0$ s.t. $D(w_0; r) \subset g(\Omega)$. Therefore, $D(-w_0; r) \cap g(\Omega) = \phi$. (see)

i.e. $|g(z) + w_0| \geq r \quad \forall\ z \in \Omega$.

Let $h(z) = \dfrac{r/3}{g(z) + w_0} : \Omega \to \mathbb{C}$. h is again univalent and $|h(z)| \leq \frac{1}{3} < 1$.

Set $\beta(z) := h(z) - h(z_0)$ $\quad$ ($z_0 \in \Omega$ arbitrarily chosen)

Then $|\beta(z)| \leq \frac{2}{3} < 1$. Hence $\beta : \Omega \to \mathbb{D}$ is univalent, holomorphic, and $0 \in \Omega_0 = \beta(\Omega)$ as we wanted.

Final Step: Consider $\mathcal{F} = \left\{ f : \Omega_0 \to \mathbb{D} \text{, holomorphic, univalent, } f(0) = 0 \right\}$.

Choose $a_0 \in \Omega_0$ and define $A = \sup\limits_{f \in \mathcal{F}} |f(a_0)|$ $\quad$ $(0 < A \leq 1)$.

Let $(f_n) \subset \mathcal{F}$ be a sequence s.t $\lim\limits_{n \to \infty} |f_n(a_0)| = A$.

By Montel's Theorem (note: $|f(z)| < 1$ $\forall z \in \Omega_0$, $f \in \mathcal{F}$) we have a uniformly convergent subseq $(f_{n_k})_{k \geq 1}$, therefore, $f = \lim\limits_{k \to \infty} f_{n_k}$ is again holomorphic (Weierstrass); $|f(a_0)| = A > 0$ and $f(0) = 0$ $\Rightarrow$ f is not constant. Hence f is univalent (Hurwitz).

Claim: $f(\Omega_0) = \mathbb{D}$. If not, let $\mathcal{K} : f(\Omega_0) \to \mathbb{D}$ be Koebe's stretching function Then $g = \mathcal{K} \circ f : \Omega_0 \to \mathbb{D}$ is in $\mathcal{F}$ and

$|g(a_0)| = |\mathcal{K}(f(a_0))| > |f(a_0)| = A$ contradiction

$\square$