Lecture 45

(45.0) Recall that last time we proved Riemann mapping theorem - using Hurwitz and Montel theorems.

<u>Riemann mapping theorem</u> - Let $\Omega \subsetneq \mathbb{C}$ be a proper, open, connected and simply-connected set; $z_0 \in \Omega$. Then $\exists!$ bijective holomorphic $f : \Omega \to \mathbb{D}$ such that $f(z_0) = 0$ and $f'(z_0) \in \mathbb{R}_{>0}$.

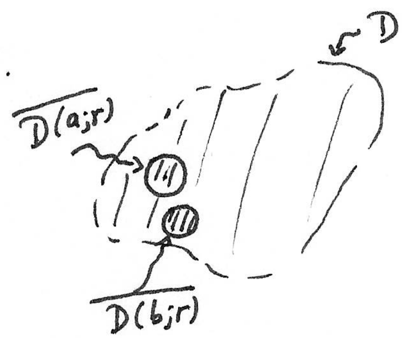
(45.1) <u>Hurwitz' Theorem</u>. - Let $D \subset \mathbb{C}$ be an <u>open, connected</u> set and $(f_n : D \to \mathbb{C})_{n \geq 1}$ a <u>uniformly convergent</u> (rel. to cpct. sets in D) sequence of holomorphic functions. Let $f = \lim_{n \to \infty} f_n : D \to \mathbb{C}$ (again holomorphic by Weierstrass' Theorem - see Lecture 19).

Assume that each f_n is univalent (i.e., 1-1) and f is not constant. Then f is univalent.

Proof. - Assume, for the sake of a contradiction, that f is not univalent.

Then $\exists\ a \neq b$; $a, b \in D$ s.t. $f(a) = f(b) = w_0$.

Let $r > 0$ be s.t. $\overline{D(a;r)}$ and $\overline{D(b;r)}$ are in D and $D(a;r) \cap D(b;r) = \phi$ (e.g. $r < \frac{|a-b|}{2}$).



Now f is not constant in $D(a;r)$ or $D(b;r)$ - by identity theorem.

Moreover $f(z) = w_0$ has at least one solution in $D(a;r)$ and in $D(b;r)$. So, for $C = C(a;r)$ or $C(b;r)$

$\{z \mid |z-a| = r\}$ $\qquad$ $\{z \mid |z-b| = r\}$

$$1 \leq \underbrace{\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)-w_0}\,dz}_{\text{counts } \#\{z\in \text{Interior}(C)\,:\, f(z)=w_0\}} = \lim_{n\to\infty} \frac{1}{2\pi i}\int_C \frac{f_n'(z)}{f_n(z)-w_0}\,dz$$

$\Rightarrow \exists\, N$ s.t. $f_N(z) = w_0$ has at least one solution in $D(a;r)$ and at least one solution in $D(b;r)$.

As the discs $D(a;r)$ and $D(b;r)$ are disjoint, this contradicts the univalence of f_N. $\square$

Remark. – The proof given above can be easily modified – or one can use Rouché's Theorem – to prove that: dropping the hypothesis of univalence, $\forall\, a\in D$, $w_0 \in \mathbb{C}$, there exists $R>0$ and $N\in\mathbb{Z}_{\geq 1}$ so that for every $n\geq N$; $f(z)=w_0$ and $f_n(z)=w_0$ have the same number of solutions in $D(a;R)$.

(45.2) Montel's Theorem. – Let $D\subset\mathbb{C}$ be an open set; $\mathcal{F}$ be a set of holomorphic functions $D\to\mathbb{C}$ satisfying:

(Local Boundedness): for every compact set $K\subset D$, there exists a constant $M_K>0$ (depending on K) such that
$$|f(z)| < M_K \; ; \text{ for every } z\in K \text{ and } f\in\mathcal{F}.$$

Then every sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{F}$ has a subsequence $(f_{n_k})_{k\geq 1}$ which converges uniformly, rel. to cpct sets in D.

Remarks. - (1) A set $\mathcal{F} \subset$ Holomorphic fns $(D \to \mathbb{C})$ satisfying the conclusion of Montel's Theorem is called a "normal family of holomorphic functions" in many classical texts on complex function theory. (A more modern term is "relatively compact").

(2). There is a way to define a metric (or distance) on the set $\mathcal{H}(D;\mathbb{C})$ = all holomorphic functions $D \to \mathbb{C}$ such that

Local Boundedness of $\mathcal{F} \subset \mathcal{H}(D;\mathbb{C})$ $\rightsquigarrow$ bounded

Normality of $\mathcal{F} \subset \mathcal{H}(D;\mathbb{C})$ $\rightsquigarrow$ closure of $\mathcal{F}$ is compact

and Montel's Theorem becomes the analogue of classical Bolzano-Weierstrass theorem [see, e.g; Conway - Functions of one complex variable - Ch. VII].

(3) Montel's Theorem is in fact an <u>if and only if</u> statement. We only need the one way implication stated above. The other direction is actually much easier - left as an exercise if you are interested.

(4) The proof of Montel's theorem - usually given in text books - is based on a much more general and classical result from real analysis - called <u>Arzelà - Ascoli Theorem</u> - stated below without a proof. I have given a "direct proof" of Montel's Thm in these notes - copying the proof of Arzelà-Ascoli result in our context (only in order to avoid more notations & definitions).

(45.3)* Arzelà - Ascoli Theorem. – <u>statement only</u>.

[Cesare Arzelà 1847-1912
Giulio Ascoli 1843-1896]

see - e.g. - Stromberg's *An introduction to classical real analysis* - Ch 3, Thm 3.145.

X : topological space having a <u>countable dense subset</u>.

Y : a <u>complete</u> metric space., say with distance function $d : Y \times Y \to \mathbb{R}_{\geq 0}$

$C(X,Y)$ = set of all continuous maps $X \to Y$.

Definition : A set $\mathcal{F} \subset C(X,Y)$ is said to be <u>equicontinuous</u> if given $x \in X$ and $\varepsilon > 0$, there is an open set $U_x \subset X$ containing x such that

$$d(f(x), f(x')) < \varepsilon \; ; \text{ for every } x' \in U_x ; \; f \in \mathcal{F}.$$

Theorem. - The following two assertions are equivalent - for $\mathcal{F} \subset C(X,Y)$.

(1) $\mathcal{F}$ is equicontinuous ; and $\forall x \in X$, the closure of $\{f(x) : x \in X\}$ in Y is compact.

(2) Every sequence $(f_n) \subset \mathcal{F}$ has a uniformly convergent subsequence.

(45.4) One of the crucial ingredients in the proof of Arzelà-Ascoli (and hence Montel's) Theorem is Cantor's famous <u>diagonal trick</u> - which I have written below as a separate lemma.

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* even more optional than the material of these notes

Lemma (Diagonal-sequence trick). Let $I = \{x_1, x_2, \dots\}$ be a countable set and $(\tilde{f}_n : I \to \mathbb{C})_{n \in \mathbb{N}}$ be a sequence of functions such that for every $x \in I$, $\{\tilde{f}_n(x)\}_{n \in \mathbb{N}}$ is bounded.

Then there exists a subsequence $(\tilde{f}_{n_k})_{k \in \mathbb{N}}$ s.t. $\lim\limits_{k \to \infty} \tilde{f}_{n_k}(x)$ exists, for every $x \in I$.

[See Stromberg - Chapter 3 - Thm. 3.144.] $(\mathbb{N} = \mathbb{Z}_{\geq 1})$

Proof. - We successively define countable infinite sets J_k $(k \geq 1)$:

$$\mathbb{N} \supset J_1 \supset J_2 \supset \cdots$$

such that $\lim\limits_{\substack{n \to \infty \\ n \in J_k}} \tilde{f}_n(x_k)$ exists $(\forall\, k \geq 1)$ and then

Choose $n_1 < n_2 < \cdots$ such that $n_k \in J_k$. Then $(\tilde{f}_{n_k})$ is the subsequence claimed to exist in this lemma.

- $k = 1$: since $\{\tilde{f}_n(x_1)\} \subset$ a compact set in $\mathbb{C}$, by Bolzano-Weierstrass, there exists a convergent subsequence - i.e. a countable infinite $J_1 \subset \mathbb{N}$ s.t. $\lim\limits_{\substack{n \to \infty \\ n \in J_1}} \tilde{f}_n(x_1)$ exists.

Assuming $J_1 \supset \cdots \supset J_k$ have been chosen, consider the set $\{\tilde{f}_n(x_{k+1}) : n \in J_k\}$ and apply Bolzano-Weierstrass again to get $J_{k+1} \subset J_k$ s.t. $\lim\limits_{\substack{n \to \infty \\ n \in J_{k+1}}} \tilde{f}_n(x_{k+1})$ exists. $\square$

(45.5) Proof of Montel's Theorem. (45.1) : I.

Let $\mathcal{F}$ be a set of holomorphic functions $D \to \mathbb{C}$, as in the statement of Theorem (45.1). Given any $f \in \mathcal{F}$, ~~&~~ $a \in D$, let us pick $r > 0$ so that $\overline{D(a;r)} \subset D$.

$$f(z) = \sum_{n=0}^{\infty} c_n(f) \cdot (z-a)^n$$ Taylor Series of f in $D(a;r)$

Recall - by Taylor's Theorem - $c_n(f) = \dfrac{1}{2\pi i} \displaystyle\int_{C(a;r)} \frac{f(z)}{(z-a)^{n+1}}\, dz.$

$$\Longrightarrow \quad \boxed{|c_n(f)| \leq \frac{M}{r^n}} \quad (*) \quad \left(M = M_{C(a;r)} \text{ - from local boundedness hypothesis} \right)$$

Now, with $a \in D$ and $r > 0$ as above, and $M = M_{C(a;r)}$, let $(f_n)_{n \in \mathbb{N}} \subset \mathcal{F}$ be a given sequence.

Apply Lemma (45.4) to : $I = \{x_0, x_1, \ldots\}$; $\left(\tilde{f}_n(x_k) = c_k(f_n)\right)_{n \geq 1}$ (boundedness assumption holds by $(*)$)

to get a subsequence $(f_{n_m})_{m \geq 1}$ s.t.

$$\lim_{m \to \infty} c_k(f_{n_m}) \text{ exists for every } k \geq 0.$$

Claim: $(f_{n_m})_{m \geq 1}$ converges uniformly on $\overline{D(a;\frac{r}{2})}$.

Proof of the claim. - given $\varepsilon > 0$; choose $N > 0$ s.t. $\frac{M}{2^{N-1}} < \frac{\varepsilon}{2}$

and k_0 s.t. $\sum_{j=0}^{N} |c_j(f_{n_k}) - c_j(f_{n_\ell})| r^j < \frac{\varepsilon}{2}$. $\forall k, \ell \geq k_0$.

Then for every $k, \ell \geq k_0$, $z \in \overline{D(a; \frac{r}{2})}$ we have :

$$|f_{n_k}(z) - f_{n_\ell}(z)| \leq \sum_{j=0}^{N} |c_j(f_{n_k}) - c_j(f_{n_\ell})| \cdot |z-a|^j + \sum_{j \geq N+1} |c_j(f_{n_k}) - c_j(f_{n_\ell})| \, |z-a|^j$$

$$\leq \frac{\varepsilon}{2} + \sum_{j=N+1}^{\infty} \frac{2M}{r^j} \frac{r^j}{2^j} = \frac{\varepsilon}{2} + \frac{M}{2^{N-1}} < \varepsilon.$$

(45.6) Proof of Montel's Theorem - II. We have shown - in the last paragraph - that for any $a \in D$, $\exists\, r_a > 0$ and a subsequence $(f_n)_{n \in J_a}$ ($J_a \subset \mathbb{N}$ infinite) converging uniformly on $\overline{D(a; r_a)}$.

→ (2nd crucial ingredient - existence of <u>a countable dense set</u> in D - e.g. take $\{a_1, a_2, \ldots\} = \{z \in D : \mathrm{Re}(z), \mathrm{Im}(z) \in \mathbb{Q}\}$.)

Let $D_j = D(a_j; r_{a_j})$. Then $D = \bigcup_{j=1}^{\infty} D_j$.

Again we use the diagonal trick. Let $J_1 \subset \mathbb{N}$ be infinite set s.t. $(f_n)_{n \in J_1}$ converges uniformly on $\overline{D_1}$.

- $J_2 \subset J_1$ s.t. $(f_n)_{n \in J_2}$ converges uniformly on $\overline{D_2}$.

$\rightsquigarrow$ we (again) get $\mathbb{N} \supset J_1 \supset J_2 \supset \dots$ $|J_k| = \infty$ and

$(f_n)_{n \in J_k}$ converges uniformly on $\overline{D_k}$.

Take $n_1 < n_2 < \dots$ such that $n_k \in J_k$ $(\forall\, k \geq 1)$.

<u>Claim</u> : $(f_{n_k})_{k \geq 1}$ converges uniformly (rel. to cpct sets) on D.

Proof. – Let $K \subset D$ be a compact set and $\varepsilon > 0$ be given.

As $K \subset \bigcup_{j=1}^{\infty} D_j$ we can find N s.t. $K \subset D_1 \cup \dots \cup D_N$

(cpct : every open cover has a finite subcover).

By construction, $(f_{n_m})_{m \geq N}$ converges uniformly on $\overline{D_1} \cup \dots \cup \overline{D_N}$

– i.e. $\exists\ \ell \geq N$ s.t. $|f_{n_{k_1}}(z) - f_{n_{k_2}}(z)| < \varepsilon$ $\forall\, k_1, k_2 > \ell$.

$\forall\, z \in \bigcup_{j=1}^{N} \overline{D_j}$

(hence $\forall\, z \in K$).

$\square$