Lecture 46

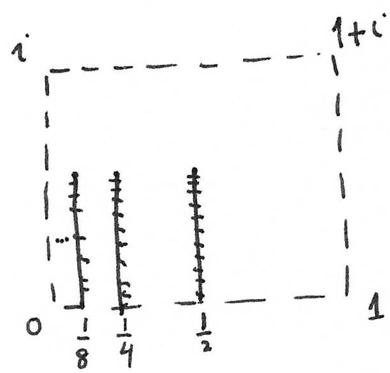
(46.0) Recall that we proved Riemann mapping theorem:

If $\Omega \subsetneq \mathbb{C}$ is a proper, open, connected and simply connected set, then there exists a conformal equivalence $f : \Omega \xrightarrow{\sim} \mathbb{D}$. (Theorem 44.0)

In other words, any two (proper, open, connected simply connected sets) Ω_1 and Ω_2 are conformally equivalent - ie., $\exists\ f : \Omega_1 \xrightarrow{\sim} \Omega_2$ (often referred to as "Riemann map".

Now we will discuss the behaviour of f at the boundary. In general, not much can be said, since in the generality of the theorem above, the relevant regions may have complicated boundaries.

e.g $$\Omega = \left\{ z \in \mathbb{C} \;\middle|\; 0 < \mathrm{Re}(z), \mathrm{Im}(z) < 1 \right\} \setminus \bigcup_{n=1}^{\infty} \left\{ \frac{1}{2^n} + it : 0 < t \leq \frac{1}{2} \right\}$$



Definition. - A point $z_0 \in \partial\Omega$ is said to be accessible if $\exists\ r_0 > 0$ such that for every $r \in (0, r_0)$ the intersection

$$\Omega \cap C(z_0; r)$$ is an arc - i.e. we have $\theta_1 < \theta_2$ (depending on r) so that $\Omega \cap C(z_0; r) = \left\{ z_0 + re^{i\theta} : \theta_1 < \theta < \theta_2 \right\}$

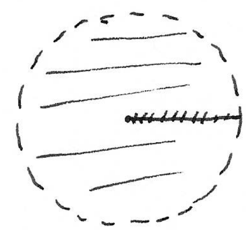
(46.1) Examples. - (1) Ω = open disc $\Rightarrow \partial\Omega$ is a circle and all points of $\partial\Omega$ are accessible.

(2) Ω = interior of a polygon (e.g. $\Omega =$ ). Again every point of $\partial\Omega$ is accessible.

(3) In general, if C is a contour (piecewise differentiable, simple, closed curve) and Ω = Interior (C) - then every point of $\partial\Omega = C$ is accessible.

(4) Cuts are not accessible - e.g. $\Omega = D(0;1) \setminus \{t : 0 \leq t < 1\}$

The points $\{t : 0 \leq t \leq 1\} \subset \partial\Omega$ are not accessible.

$\Omega =$ 

(46.2) Let $\Omega_1, \Omega_2 \subsetneq \mathbb{C}$ be open, connected, simply-connected and bounded subsets of $\mathbb{C}$ s.t. every point of $\partial\Omega_j$ is accessible.

Let $f : \Omega_1 \xrightarrow{\sim} \Omega_2$ be a conformal equivalence.

Theorem. - f extends to a continuous function $\tilde{f} : \overline{\Omega}_1 \to \overline{\Omega}_2$ which admits a continuous inverse (i.e. - $\tilde{f}$ is a <u>homeomorphism</u>)

As stated, this theorem is a weaker version of a result due to Caratheodory ; Osgood-Taylor (1913) which states that the conformal equivalence $f : \Omega \to \mathbb{D}$ extends to the boundary $\partial\Omega \iff \partial\Omega$ is a Jordan curve (i.e. continuous, simple, closed curve).

The weaker version (Theorem 46.2) will however be sufficient for our purposes.

(46.3) <u>Proof of Theorem 46.2.</u> We will show that $\forall z_0 \in \partial\Omega_1$,

$\lim\limits_{\substack{z \to z_0 \\ z \in \Omega_1}} f(z)$ exists. By Problem 12 of Set 2, this will imply that

$$\tilde{f} : \overline{\Omega}_1 \to \overline{\Omega}_2 \quad \text{defined by} \quad \tilde{f}(z_0) = \begin{cases} f(z_0) & \text{if } z_0 \in \Omega_1 \\ \lim\limits_{\substack{z \to z_0 \\ z \in \Omega}} f(z) & \text{if } z_0 \in \partial\Omega_1 \end{cases}$$

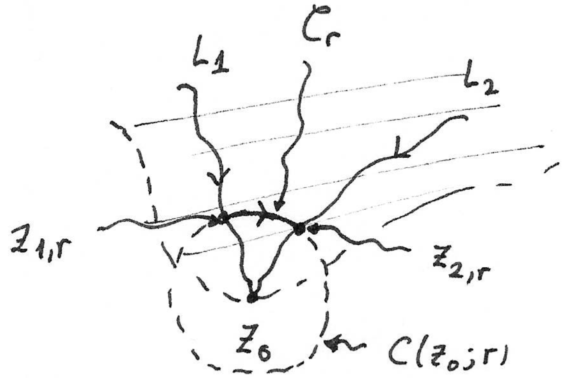
is continuous.

Carrying out the same argument for $g = f^{-1} : \Omega_2 \to \Omega_1$, we will obtain continuous functions $\overline{\Omega}_1 \underset{\tilde{g}}{\overset{\tilde{f}}{\rightleftarrows}} \overline{\Omega}_2$ s.t. $\tilde{f} \circ \tilde{g} =$ identity on Ω_2
$\tilde{g} \circ \tilde{f} =$ identity on Ω_1

and hence, by uniqueness of extending a continuous functions $\tilde{g} = \tilde{f}^{-1}$.

In conclusion, it suffices to show that $\lim\limits_{\substack{z \to z_0 \\ z \in \Omega_1}} f(z)$ exists, $\forall z_0 \in \partial\Omega_i$

Assume, on the contrary, that



$\lim_{\substack{z \to z_0 \\ z \in \Omega_1}} f(z)$ does not exist. Then,

we can find continuous paths

$L_1, L_2 : [0,1) \to \Omega$; $L_j(1) = z_0$; so that

$$\delta = \left| \lim_{\substack{z \in L_1 \\ z \to z_0}} f(z) - \lim_{\substack{z \in L_2 \\ z \to z_0}} f(z) \right| > 0.$$

Thus, $\exists\, r_0 > 0$ s.t. for every $r \in (0, r_0)$, $\cancel{|z_{1,r} - f(z_{2,r})}$

$d_r := |f(z_{1,r}) - f(z_{2,r})| > \frac{\delta}{2}$; where $z_{1,r}$ and $z_{2,r}$ are the points on L_1, L_2 intersected with circle $C(z_0, r)$ - see picture above

By definition of "accessible" point - $z_{1,r}$ and $z_{2,r}$ can be connected by an arc - $z_0 + re^{i\theta}$ $(\theta_1 \leq \theta \leq \theta_2)$ - say C_r.

$$f(z_{2,r}) - f(z_{1,r}) = \int_{C_r} f'(z)\,dz = \int_{\theta_1}^{\theta_2} f'(z_0 + re^{i\theta})\, r.e^{i\theta}.i.d\theta$$

$$\Rightarrow d_r^2 = |f(z_{2,r}) - f(z_{1,r})|^2 = r^2 \left| \int_{\theta_1}^{\theta_2} f'(z_0 + re^{i\theta})\, i e^{i\theta} d\theta \right|^2$$

$$\leq r^2 \int_{\theta_1}^{\theta_2} |f'(z_0 + re^{i\theta})|^2 d\theta \; \boxed{\int_{\theta_1}^{\theta_2} |ie^{i\theta}|^2 d\theta} \leftarrow \leq 2\pi$$

by Schwarz' integral inequality : $\left|\int f_1 f_2\right|^2 \leq \int |f_1|^2 \cdot \int |f_2|^2$

i.e. $$\frac{\delta^2}{4r} \leq \frac{d_r^2}{r} \leq 2\pi \int_{\theta_1}^{\theta_2} |f'(z_0+re^{i\theta})|^2 \, r \, d\theta$$

integrating w.r.t. $r \in (0, r_0)$ gives L.H.S. $= \infty$.

But $$\text{R.H.S.} = 2\pi \int_0^{r_0} \int_{\theta_1}^{\theta_2} |f'(z_0+re^{i\theta})|^2 \, r \, d\theta \, dr$$

$$= 2\pi \cdot \text{Area}\left(f\left(\left\{ z_0+re^{i\theta} : \begin{smallmatrix} 0 \leq r \leq r_0 \\ \theta_1 \leq \theta \leq \theta_2 \end{smallmatrix} \right\} \right) \right)$$

$$\leq 2\pi \, \text{Area}(\Omega_2) < \infty.$$

This contradiction proves that $\lim\limits_{\substack{z \to z_0 \\ z \in \Omega_1}} f(z)$ must exist. □

(46.4) Recall. Dirichlet's boundary value and variational problems from Lecture 40 (§40.4).

For simplicity let Ω be an open, connected, simply connected bounded domain whose boundary points are accessible (as in Thm 46.2).

Boundary Value Problem. given $g : \partial\Omega \to \mathbb{R}$ cnts function, find $u \in C^2(\Omega)$ harmonic s.t. $\boxed{\lim\limits_{\substack{z \to z_0 \\ z \in \Omega}} u(z) = g(z_0) \quad \forall z_0 \in \partial\Omega. \quad (*)}$

Variational Problem given $g : \partial\Omega \to \mathbb{R}$ continuous function, find $u \in C^2(\Omega)$ s.t. $(*)$ holds and $E[u] = \iint_\Omega (u_x^2 + u_y^2) \, dx \, dy$ is min.

- In the next lecture we will solve BVP for $\Omega = \mathbb{D}$, which together with Thm 46·2 solves it for any Ω whose boundary is accessible.

Hadamard's counterexample to Dirichlet's variational principle.

Let $\Omega = \mathbb{D}$; $g : \partial\Omega = C(0;1) \to \mathbb{R}$ be given by

$$g(\theta) = \sum_{n=1}^{\infty} \frac{\cos(n!\,\theta)}{n^2}. \quad \text{Then } u(re^{i\theta}) = \sum_{n=1}^{\infty} \frac{r^{n!}\cos(n!\,\theta)}{n^2}$$

$$= \mathrm{Re}\left(\sum_{n=1}^{\infty} \frac{z^{n!}}{n^2}\right) \text{ solves Dirichlet's BVP, but}$$

$$E[u] = \pi \sum_{n=1}^{\infty} \frac{n!}{n^4} \quad \text{(by the "area formula" - § 39.4)}$$

$$= \infty. \quad \text{i.e. Variational Problem is unsolvable.}$$