Lecture 47

(47.0) Recall that we proved in Lecture 46 - (Riemann maps at the boundary)

- Let Ω_1 and Ω_2 be two subsets of $\mathbb{C}$ which are interiors of two contours C_1, C_2.

$$\Omega_j = \text{Interior}(C_j) \; ; \; j = 1, 2.$$

Let $f : \Omega_1 \xrightarrow{\sim} \Omega_2$ be a conformal equivalence. (obtained from Riemann mapping theorem $\Omega_1 \xrightarrow{\sim} \mathbb{D} \xleftarrow{\sim} \Omega_2$.) Then f extends to a continuous $\tilde{f} : \overline{\Omega}_1 \to \overline{\Omega}_2$ (closures of Ω_j's : $\overline{\Omega}_j = \Omega_j \cup C_j$) admitting a continuous inverse (by switching the rôle of Ω_1, Ω_2).

This theorem allows us to reduce Dirichlet's Boundary Value Problem for Ω = Interior of a contour C; to the unit disc $\mathbb{D}$.

(47.1) <u>Solution to Dirichlet's BVP for $\mathbb{D}$</u> is obtained via <u>Poisson's kernel</u>*

<u>Poisson Kernel</u> : For $z \in \mathbb{D}$; $\phi \in \mathbb{R} \pmod{2\pi}$, define

$$P(z;\phi) := \frac{e^{i\phi} + z}{e^{i\phi} - z}$$

Let $$K_p(z;\phi) = \mathrm{Re}\,(P(z;\phi)) = \frac{1 - |z|^2}{|e^{i\phi} - z|^2} \quad \text{(easy check).}$$

$$= \frac{1 - r^2}{1 - 2r\cos(\theta - \phi) + r^2} \quad \text{if } z = re^{i\theta}.$$

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* Simeon Denis Poisson (21/6/1781 - 25/4/1840)

<u>Theorem</u> : Let $g : \partial\mathbb{D} \to \mathbb{R}$ be a continuous function, written as $g(\phi)$ $(\phi \in \mathbb{R})$ - so that $g(\phi + 2\pi) = g(\phi)$. Then

$$u(z) := \frac{1}{2\pi} \int_0^{2\pi} K_p(z;\phi)\, g(\phi)\, d\phi \quad : \mathbb{D} \to \mathbb{R}$$

is a harmonic function such that for every ϕ_0

$$\lim_{\substack{z \to e^{i\phi_0} \\ z \in \mathbb{D}}} u(z) = g(\phi_0).$$

Remark. - As we have seen previously (Lectures 40. §40.5 ; Lecture 46) solution to Dirichlet's BVP is unique (if exists). Thus the theorem above gives <u>the</u> solution to this BVP for $\mathbb{D}$.

(47.2) <u>Derivation of Poisson Kernel - I.</u> (using power series)

Let $u : \mathbb{D} \to \mathbb{R}$ be a harmonic function (i.e. $u_{xx} + u_{yy} = 0$).

Then $u = \mathrm{Re}(f)$ for a holomorphic function $f : \mathbb{D} \to \mathbb{C}$ (see Lecture 14, § 14.1). Recall that $f = u + iv$ is found by solving Cauchy-Riemann equations $\boxed{\begin{aligned} v_x &= -u_y \\ v_y &= u_x \end{aligned}}$ and we will put the initial condition $v(0) = 0$ (i.e. $f(0) = u(0) \in \mathbb{R}$) to fix the constant of integration.

Let $f(z) = a_0 + \sum_{n\geq 1} a_n z^n$ be the Taylor Series expansion of f near 0. $(a_0 = f(0) \in \mathbb{R})$.

$$\Rightarrow \quad u(z) = \operatorname{Re}(f(z)) = \frac{1}{2}\left(f(z) + \overline{f(z)}\right)$$

$$= a_0 + \frac{1}{2}\sum_{n\geq 1}\left(a_n z^n + \overline{a_n}\,(\overline{z})^n\right)$$

$$= a_0 + \frac{1}{2}\sum_{n\geq 1} r^n\left(a_n e^{in\theta} + \overline{a_n}\, e^{-in\theta}\right) \; ; \text{ if } z = re^{i\theta}.$$

Using $\frac{1}{2\pi}\int_0^{2\pi} e^{ik\theta}\,d\theta = \begin{cases} 0 & \text{if } k \neq 0 \\ 1 & \text{if } k = 0 \end{cases}$, we get:

$$(*) \quad a_0 = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\,d\theta \quad ; \quad a_n = \frac{1}{\pi}\int_0^{2\pi} \frac{u(re^{i\theta})\,e^{-in\theta}}{r^n}\,d\theta .$$

Note - we can take $0<r<1$ as large as we want. If u extends to a continuous $\overline{\mathbb{D}} \to \mathbb{R}$, we can set $r=1$. I will change θ to ϕ to match with previous notation.

Substituting $(*)$ in the series $f(z) = \sum_{l=0}^{\infty} a_l z^l$, we get

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\phi})\,d\phi + 2\cdot\sum_{n=1}^{\infty}\frac{1}{2\pi}\int_0^{2\pi} u(e^{i\phi})\left(\frac{z}{e^{i\phi}}\right)^n d\phi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\phi})\left\{1 + 2\sum_{n=1}^{\infty}\left(z\cdot e^{-i\phi}\right)^n\right\} d\phi$$

$\left(\int\sum = \sum\int\right.$ by uniform convergence of geometric series - Lectures 19,20$\left.\right)$

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\phi}) \left\{ 1 + \frac{2 \cdot z \cdot e^{-i\phi}}{1 - z \cdot e^{-i\phi}} \right\} d\phi$$

$$\boxed{f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\phi} + z}{e^{i\phi} - z} \, u(e^{i\phi}) \, d\phi} \qquad \forall \ z \in \mathbb{D}.$$

Taking real part of this equation gives

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{i\phi} - z|^2} \, u(e^{i\phi}) \, d\phi$$; written, equivalently

in polar form as $$u(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - \phi) + r^2} \, u(e^{i\phi}) \, d\phi.$$

[calculation of the real part of $P(z;\phi)$:

$$P(z;\phi) = \frac{e^{i\phi} + z}{e^{i\phi} - z} = \frac{(e^{i\phi} + z)(e^{-i\phi} - \bar{z})}{(e^{i\phi} - z)(e^{-i\phi} - \bar{z})}$$

$$= \frac{1 - |z|^2 + \underline{(z e^{-i\phi} - \bar{z} e^{i\phi})}}{|e^{i\phi} - z|^2}$$

$\alpha - \bar{\alpha}$ is purely imaginary

$$\Rightarrow K_p(z;\phi) = \mathrm{Re}(P(z;\phi)) = \frac{1 - |z|^2}{|e^{i\phi} - z|^2}.$$

If $z = re^{i\theta}$, the denominator becomes

$$(\cos\phi - r\cos\theta)^2 + (\sin\phi - r\sin\theta)^2$$

$$= 1 + r^2 - 2r(\cos\phi \cos\theta + \sin\phi \sin\theta)$$

$$= 1 - 2r\cos(\theta - \phi) + r^2. \quad]$$

(47.3) Derivation of $K_p(z;\phi)$ - II. using $\mathrm{Aut}(\mathbb{D})$: Möbius transf.

Recall the mean value property of harmonic functions:

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{it})\,dt \qquad \text{(Lecture 16, §16.1).}$$

Let $z_0 \in \mathbb{D}$ and $\varphi(z) = \dfrac{z+z_0}{1+\bar{z}_0 z} \in \mathrm{Aut}(\mathbb{D})$ $(\varphi(0) = z_0)$.

(Easy) Exercise: $w(z) = u(\varphi(z))$ is again harmonic

$$w(0) = \frac{1}{2\pi}\int_0^{2\pi} w(e^{it})\,dt \qquad \text{(mean value prop. of } w\text{)}$$

$$\Rightarrow \quad u(\varphi(0)) = \frac{1}{2\pi}\int_0^{2\pi} u(\varphi(e^{it}))\,dt$$

i.e. $$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u\left(\frac{e^{it}+z_0}{1+e^{it}\bar{z}_0}\right) dt.$$ Perform the change of variables

$$e^{i\phi} = \frac{e^{it}+z_0}{1+e^{it}\bar{z}_0} \rightsquigarrow e^{it} = \frac{e^{i\phi}-z_0}{1-e^{i\phi}\bar{z}_0}$$

$$\Rightarrow \quad dt = \frac{1-|z_0|^2}{|e^{i\phi}-z_0|^2}\,d\phi.$$ (Exercise)

Hence, (limits of $\int$ can be adjusted via a rotation)

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z_0|^2}{|e^{i\phi}-z_0|^2}\,u(e^{i\phi})\,d\phi.$$

(47.4) Properties of $K_p(z;\phi) = \dfrac{1-|z|^2}{|e^{i\phi}-z|^2}$.

- $K_p(z;\phi) > 0 \quad \forall\, z \in \mathbb{D}$ – obviously.
- $K_p(z;\phi) = 0$ for $|z| = 1$; $z \neq e^{i\phi}$.
- $\boxed{\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} K_p(z;\phi)\, d\phi = 1}$ "Total mass = 1"

(take u = constant fn. 1 in $u(\overset{z}{\cancel{re^{i\theta}}}) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} K_p(z;\phi)\, u(e^{i\phi})\, d\phi$)

In conclusion: $K_p(z;\phi)$ defines a "probability distribution" on $\partial\mathbb{D}$, and as $z \to \partial\mathbb{D}$, it is concentrated near $e^{i\phi}$. (Dirac δ-style).

(47.5) <u>Proof of Theorem 47.1</u> (see page 2 above)

Given $g(\theta)$ (cnts $\partial\mathbb{D} \to \mathbb{R}$) define $u(z) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} K_p(z;\phi)\, g(\phi)\, d\phi$.

As $u = \mathrm{Re}\left(\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \dfrac{e^{i\phi}+z}{e^{i\phi}-z}\, g(\phi)\, d\phi \right)$ = real part of a hol. fn.

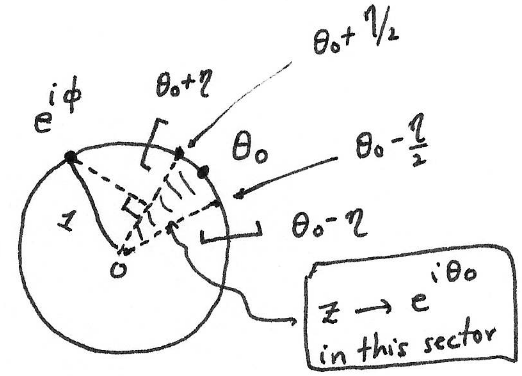
u is harmonic

(Ex: verify this directly.)

It remains to show: $\displaystyle\lim_{\substack{r \to 1^- \\ \theta \to \theta_0}} u(re^{i\theta}) = g(\theta_0)$.

Note: $$u(re^{i\theta}) - g(\theta_0) = \frac{1}{2\pi}\int_0^{2\pi} K_p(re^{i\theta};\phi)\,\big(g(\phi)-g(\theta_0)\big)\,d\phi$$

because $$\frac{1}{2\pi}\int_0^{2\pi} K_p(re^{i\theta};\phi)\,d\phi = 1 \quad (\text{"Total mass} = 1\text{"}).$$



Let $\varepsilon>0$ be given.

By continuity of g, there exists $\eta>0$ so that $$|g(\phi)-g(\theta_0)| < \frac{\varepsilon}{2}$$ for every ϕ so that $|\phi-\theta_0|<\eta$. Write

$$\frac{1}{2\pi}\int_0^{2\pi} = \frac{1}{2\pi}\int_{|\phi-\theta_0|<\eta} + \frac{1}{2\pi}\int_{|\phi-\theta_0|\geq\eta}$$

- $$\left|\frac{1}{2\pi}\int_{|\phi-\theta_0|<\eta} K_p(z;\phi)\,\big(g(\phi)-g(\theta_0)\big)\,d\phi\right|$$
$$\leq \frac{\varepsilon}{2}\cdot\left(\frac{1}{2\pi}\int_0^{2\pi} K_p(z;\phi)\,d\phi\right) \quad \text{by positivity of } K_p(z;\phi)$$
$$= \frac{\varepsilon}{2} \quad (\text{"Total mass} = 1\text{"})$$

- for ϕ s.t. $|\phi-\theta_0|\geq\eta$: as $z=re^{i\theta}\to e^{i\theta_0}$; we may restrict to $|\theta-\theta_0|<\frac{\eta}{2}$. For this range, $|\theta-\phi|\geq\frac{\eta}{2}$ and hence $|re^{i\theta}-e^{i\phi}|\geq\sin\left(\frac{\eta}{2}\right)$ (see picture above)

$$\Rightarrow \left|\frac{1}{2\pi}\int_{|\phi-\theta_0|\geq\eta} K_p(re^{i\theta};\phi)\,\big(g(\phi)-g(\theta_0)\big)\,d\phi\right| \leq \frac{M\cdot(1-r^2)}{\sin\left(\frac{\eta}{2}\right)}$$

where $\displaystyle M = \operatorname*{Max}_{0 \leq \phi \leq 2\pi} |g(\phi) - g(\theta_0)|$.

Since $r \to 1$; we can make $1 - r_0^2 < \dfrac{\sin(\eta/2)}{M} \cdot \dfrac{\varepsilon}{2}$. for some $r_0 \in (0,1)$

Combining all this, we conclude that - given $\varepsilon > 0$, there exists $\eta > 0$ and $r_0 > 0$ such that - for $z = re^{i\theta}$, $|\theta - \theta_0| < \frac{\eta}{2}$, $r_0 < r < 1$

$$\left| \frac{1}{2\pi} \int_0^{2\pi} K_p(re^{i\theta}, \phi)\, g(\phi)\, d\phi \;-\; g(\theta_0) \right| < \varepsilon$$

i.e. $\displaystyle \lim_{\substack{r \to 1^- \\ \theta \to \theta_0}} u(re^{i\theta}) = g(\theta_0)$ as claimed. $\square$