Lecture 49

(49.0) <u>Doubly-periodic functions.</u> (<u>also called elliptic functions</u>)

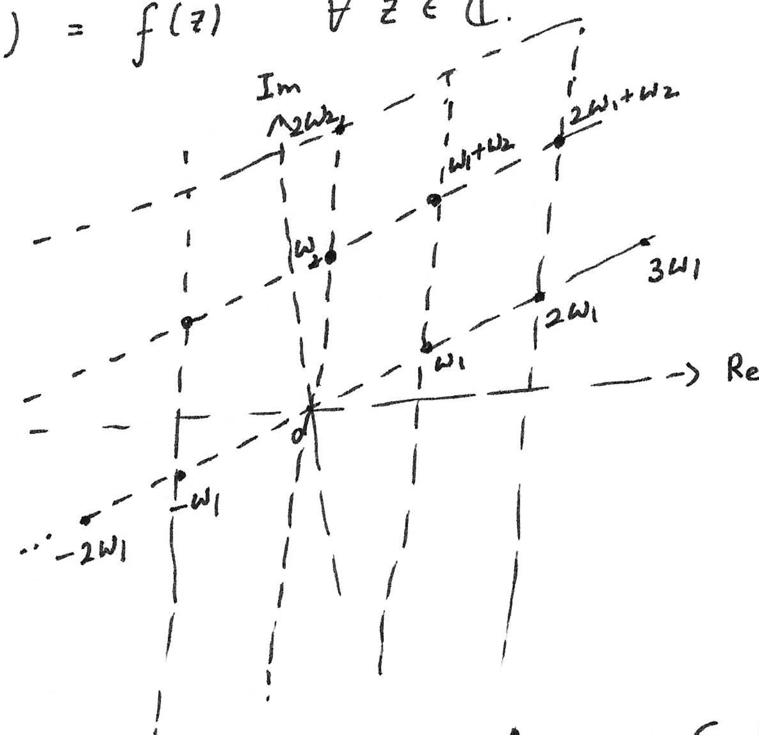
Let $\omega_1, \omega_2 \in \mathbb{C} \setminus \{0\}$ be such that $\frac{\omega_2}{\omega_1} \notin \mathbb{R}$. (e.g. $\omega_1 = 1$; $\omega_2 = \tau \in \mathbb{H}$)

$\Lambda := \{ m\omega_1 + n\omega_2 : m, n \in \mathbb{Z} \} \subset \mathbb{C}$ called <u>period lattice</u>.

A <u>doubly-periodic function</u> is $f : \mathbb{C} \to \mathbb{C}$ meromorphic s.t.

$$f(z+\gamma) = f(z) \quad \forall z \in \mathbb{C}.$$

i.e. $$\boxed{\begin{array}{c} f(z+\omega_j) = f(z) \\ (j = 1, 2) \end{array}}$$



Points of $\Lambda_{(\omega_1, \omega_2)} \subset \mathbb{C}$.

Analogy with "singly-periodic" or trigonometric functions

$$e^{z + 2\pi n i} = e^{z}$$

$$\forall n \in \mathbb{Z}$$

In general $g(z+1) = g(z)$ means $g(z)$ is in fact a function of $e^{2\pi i z}$. Therefore, any entire function $F(z) \rightsquigarrow F(e^{2\pi i z})$ 1-periodic entire function.

Our first main result is about the impossibility of entire doubly-periodic functions.

(49.1) **Prop.**- Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic and elliptic. Then f is a constant.

<u>Proof</u>. Let Λ be the periodic lattice of f. Then, $|f(z)|$ is bounded by $M = \mathrm{Max}\{|f(z)| : z \in \overline{P_0}\}$

P_0 = (open) rectangular region with vertices $\{0, \omega_1, \omega_2, \omega_1+\omega_2\}$

Note: Let

$D_0 = \{ s\omega_1 + t\omega_2 : 0 \leq s,t \leq 1\}$

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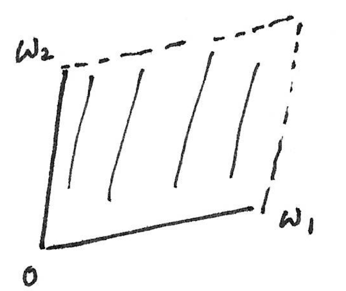


. By $z_1 \equiv z_2 \mod \Lambda$, we mean $z_1 - z_2 \in \Lambda$.

(1) $f(z_1) = f(z_2)$ for every $z_1 \equiv z_2 \mod \Lambda$

(2) For every $z \in \mathbb{C}$, there is a unique $z_0 \in D_0$ s.t. $z \equiv z_0 \mod \Lambda$.

Hence f is entire and bounded, which implies f is constant, by Liouville's Theorem. □

(49.2) Remark.- The proposition above gives a method of proving identities among elliptic functions.

- show that they have the same periodic lattice
- they ~~are~~ have same set of poles - and same singular behaviour

(49.2) <u>Prop.</u> Let $f : \mathbb{C} \to \mathbb{C}$ be an elliptic function with period lattice Λ. Let $D_t = \{ t + s_1 \omega_1 + s_2 \omega_2 \mid s_1, s_2 \in [0,1) \}$.

Let $\begin{cases} \alpha_1, \alpha_2, \dots, \alpha_m : & \text{~~poles~~ zeroes of } f \text{ - listed with mult.} \\ \beta_1, \beta_2, \dots, \beta_n : & \text{poles of } f \text{ - listed according to their order.} \end{cases}$

necessarily finite - see remark on page 5 below

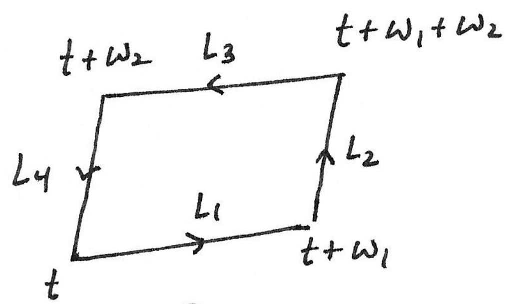
(t so that f does not have poles on ∂D_t)

$\alpha_1, \dots, \alpha_m ; \beta_1, \dots, \beta_n \in D_t$. Then:

(1) $$\sum_{j=1}^{n} \operatorname*{Res}_{z=\beta_j} f(z) = 0.$$

(2) $m = n$ (i.e. # zeroes within $(C_t = \partial D_t)$ = # poles within C_t)

(3) $$\sum_{j=1}^{n} \alpha_j - \sum_{j=1}^{n} \beta_j \in 2\pi i \Lambda.$$



C_t

Proof. (1)

$$\frac{1}{2\pi i} \int_{C_t} f(z)\,dz = \sum_{j=1}^{n} \operatorname*{Res}_{z=\beta_j} (f(z))$$

But
$$\int_{C_t} f(z)\,dz = \int_{L_1} + \int_{L_3} + \int_{L_2} + \int_{L_4} f(z)\,dz$$

$$= \int_t^{t+\omega_1} f(z)\,dz - \int_{t+\omega_2}^{t+\omega_2+\omega_1} f(z)\,dz - \int_t^{t+\omega_2} f(z)\,dz + \int_{t+\omega_1}^{t+\omega_1+\omega_2} f(z)\,dz$$

- change variables: $z \pm \omega_2 = s$ in the second integral and $z - \omega_1 = s$ in the fourth.

Using $f(s+\omega_j) = f(s)$, we get $\displaystyle\int_{C_t} f(z)\,dz = 0$.

(2) Repeat the same argument as above, for $\dfrac{f'(z)}{f(z)}$.

$$\frac{1}{2\pi i}\int_{C_t} \frac{f'(z)}{f(z)}\,dz = \#\text{ Zeroes within } C_t - \#\text{ Poles within } C_t$$

$$= m - n \quad \text{(in general)}$$

L.H.S $= 0$ for periodicity reasons.

(3) Finally $\displaystyle\frac{1}{2\pi i}\int_{C_t} z\cdot\frac{f'(z)}{f(z)}\,dz = \sum_{j=1}^{n}\alpha_j - \sum_{k=1}^{n}\beta_k$ in general

But
$$\int_{C_t} z\cdot\frac{f'(z)}{f(z)}\,dz = \int_t^{t+\omega_1} z\cdot\frac{f'(z)}{f(z)} - (z+\omega_2)\frac{f'(z)}{f(z)}\,dz$$

$$- \int_t^{t+\omega_2} z\cdot\frac{f'(z)}{f(z)} - (z+\omega_1)\frac{f'(z)}{f(z)}\,dz$$

$= -\omega_2 \cdot \left(\text{Winding number of } \{f(t+s\omega_1) : s \in [0,1]\} \text{ around } 0\right) \cdot 2\pi i$

$+\ \omega_1 \cdot \left(\text{ " \quad " \quad " } \{f(t+s\omega_2) : s \in [0,1]\} \text{ around } 0\right) \cdot 2\pi i$

(reason : $\displaystyle\int_t^{t+\omega_1} \frac{f'(z)}{f(z)}\,dz = \int_\gamma \frac{du}{u} \in 2\pi i\,\mathbb{Z}.$)

$$\gamma = \{ f(t+s\omega_1) : s \in [0,1] \}$$

$\Rightarrow$ ~~# zeros~~ $\displaystyle\sum_{j=1}^n \alpha_j - \sum_{k=1}^n \beta_k \in 2\pi i\,\mathbb{Z}.$ $\square$

(49.3) Remark.- If $f : \mathbb{C} \dashrightarrow \mathbb{C}$ is a mero. elliptic function $P \subset \mathbb{C}$ its set of poles (recall: meromorphic means all (non-removable) singularities are poles. Then P is discrete - i.e. every point of P is isolated $\Rightarrow \overline{D}_t \cap P$ is finite being a discrete subset of a compact set.

(49.4) Examples. - (1) Historically speaking, the first example of an elliptic function was given by Abel (1827)

$$u = \mathrm{sn}(z\,;k) \quad \text{means} \quad \frac{du}{dz} = \sqrt{1-u^2}\,\sqrt{1-k^2u^2}$$

(normalize : $u(0)=0$.)

(here k is usually taken to be in $(0,1)$)

(though the "inverse" of elliptic functions - known as elliptic integrals - had been studied by Wallis (1655), Newton (1695)

Fargano (1718), Euler (Dec. 23, 1751 – "birthday of elliptic functions" – Jacobi)

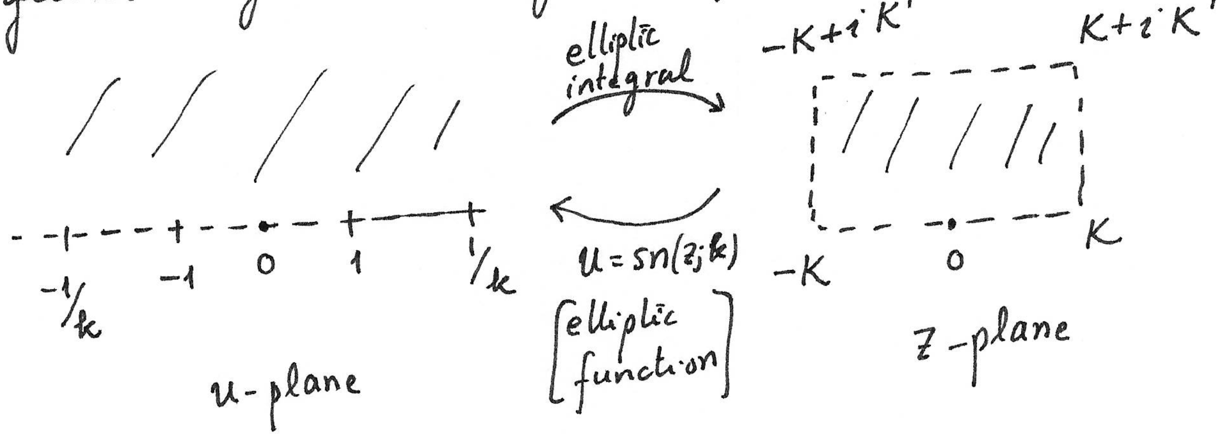
Legendre (1825)).

$$\frac{du}{dz} = \sqrt{1-u^2}\cdot\sqrt{1-k^2u^2} \Rightarrow dz = \frac{du}{\sqrt{1-u^2}\sqrt{1-k^2u^2}}$$

$$\rightsquigarrow \quad z = \int_0^u \frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}$$

$u \in \mathbb{H}$ (say) and integral is over any path joining $0 \to u$

Remark – geometrically – above integral defines a conformal equivalence



Change of variables $\quad t = \sin(\theta) \rightsquigarrow dt = \cos(\theta)\, d\theta$

$$z = \int_0^{\sin^{-1}(u)} \frac{d\theta}{\sqrt{1-k^2\sin^2(\theta)}}$$

(elliptic integral of first kind – Legendre (1825))

Jacobi's amplitude function:
$$\phi = \int_0^{am(\phi)} \frac{dt}{\sqrt{1-k^2\sin^2(t)}} \quad -(*)$$

so that $sn(z;k) = \sin(am(z))$.

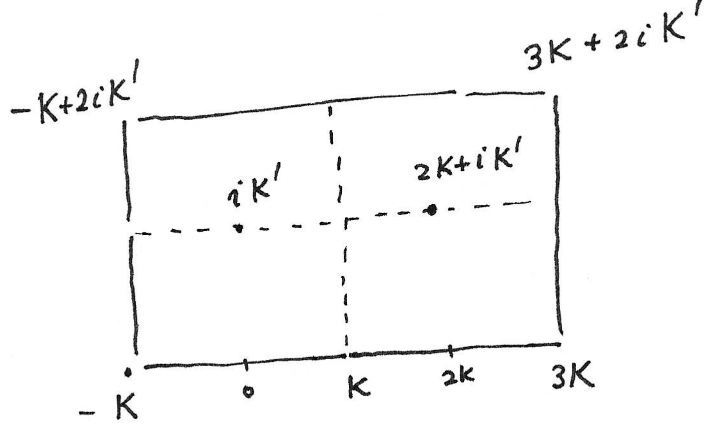
<u>Properties</u>: (1) $sn(-z;k) = -sn(z;k)$ since sine is odd and $am(-\phi) = -am(\phi)$.

[Note: taking derivative of $(*)$ w.r.t. ϕ: $1 = \frac{1}{\sqrt{1-k^2\sin^2(am\phi)}} \cdot \frac{d\,am(\phi)}{d\phi}$

i.e. $\frac{d}{d\phi} am(\phi) = \sqrt{1-k^2\sin^2(am(\phi))}$; $am(0) = 0$.]

(2) Abel (1827) obtained addition formulae for $sn(z;k)$ and used these to prove that

- $sn(z;k)$ is doubly-periodic with period lattice $\{m(4K)+n(2iK') : m,n \in \mathbb{Z}\}$
- zeroes at $0, 2K$
  poles (simple) at iK' and $2K+iK'$
  (residues $\frac{1}{k}$, $\frac{-1}{k}$ respectively)



– we will prove this result later.

(49.5) Examples - (2) Jacobi's theta function (1829)

$\theta(z;\tau)$ is defined to be the <u>unique holomorphic function</u> $\mathbb{C} \to \mathbb{C}$ satisfying (here $\tau \in \mathbb{H}$, i.e. $\mathrm{Im}(\tau) > 0$)

- $\theta(z+1;\tau) = -\theta(z;\tau)$; $\theta(z+\tau;\tau) = -e^{-2\pi i z} e^{-\pi i \tau} \theta(z;\tau)$
- $\theta(z;\tau) = 0 \iff z = m + n\tau$; $m, n \in \mathbb{Z}$
- $\theta'(0;\tau) = 1$.

Remark. - $\theta(z;\tau)$ is <u>not</u> doubly-periodic, but is fundamental in the theory of doubly-periodic functions because

(Jacobi) - Any doubly-periodic function can be written as a ratio of theta functions - providing a converse to Proposition (49.2)