Lecture 1

(1.0) Last time we introduced linear equations and set ourselves the goal of solving a system of linear equations.

Today we will formulate this goal precisely and go over a general method to simplify and solve a linear system.

(1.1) Assume that we are given $\underline{m}$ linear equations in $\underline{n}$ variables ($\underline{m \text{ and } n \text{ are fixed positive integers}}$).

In general, such a linear system can be written as:

$$(*)\left\{\begin{array}{lr}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 & -(\text{Eq}^{n}1) \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 & -(\text{Eq}^{n}2) \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3 & -(\text{Eq}^{n}3) \\
\vdots & \vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m & -(\text{Eq}^{n}m)
\end{array}\right.$$

- $a_{11}, \ldots, a_{1,n}, \ldots, a_{m1}, \ldots, a_{mn}$ are fixed numbers ($m \cdot n$ many) – called coefficients of the linear system.
- $x_1, x_2, \ldots, x_n$ are unknowns (variables).
- $b_1, b_2, \ldots, b_m$ are also fixed numbers.

(1.2) A linear system can be conveniently encoded via <u>matrices</u>.

Definition.— An <u>$(m \times n)$ matrix</u> is a <u>rectangular array of numbers</u>:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

(# of rows = m ; # of columns = n)

Such a matrix is usually abbreviated as:

$$A = (a_{\underset{\text{row index}}{i}\,\underset{\text{column index}}{j}}) \quad \begin{matrix} 1 \le i \le m \\ 1 \le j \le n \end{matrix}$$

e.g. $\begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & 9 \end{bmatrix}$ is a 2×3 matrix.

$\begin{bmatrix} 0 & 1 \\ -1 & 5 \end{bmatrix}$ is a 2×2 matrix. [square matrix].

If $m = n$, we say that the matrix is a <u>square matrix</u>.

(1.3) Representing a linear system as a matrix.

A general linear system of <u>m equations</u> in <u>n unknowns</u> (*) from page 1 above is usually written as the following matrix of <u>m rows</u> and <u>(n+1) columns</u> :

$$B = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ & \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

$\underbrace{\phantom{a_{m1}\ a_{m2}\ \cdots\ a_{mn}}}_{A}$

– called the <u>augmented matrix</u> for the system (*).

The $(m \times n)$ matrix $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ is called

the <u>coefficient matrix</u> of the system (*).

e.g. Consider the following linear system of 3 equations in 3 unknowns

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 4 \\ x_1 \qquad + x_3 &= 1 \\ x_2 - 2x_3 &= 5 \end{aligned} \quad \rightsquigarrow \quad A = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

Coefficient matrix

[Note: missing variable means corresponding coefficient is 0.]

$$B = \left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & 5 \end{array}\right]$$

Augmented matrix

(1.4) <u>Elementary Operations</u>. There are 3 operations we can perform on a linear system without changing its set of solutions.

Let us consider a linear system of m equations, abbreviated as $E_1, \dots, E_m$.

Elementary Operation I. (Swap) - We can interchange 2 equations (if we swap E_i and E_j, we will write it as $\boxed{E_i \leftrightarrow E_j}$)

Elementary Operation II. (Scale) - We can choose a non-zero number, say $\underset{\substack{\uparrow\\ \text{(alpha)}}}{\alpha}$, and replace $\underset{\substack{\uparrow\\ (i^{th}\text{ equation})}}{E_i}$ by $\alpha \cdot E_i$. $\boxed{E_i \to \alpha E_i}$

Elementary Operation III (Combine) - We can replace E_i by $E_i + \lambda E_j$, where $j \neq i$ and λ is an arbitrary number.

$\boxed{E_i \longrightarrow E_i + \lambda E_j}$.

Theorem 1. Elementary operations do not change the set of solutions of the linear system.

Proof.* - It is clear that I and II can be reversed. Let us see why III can also be reversed.

$$\boxed{\begin{array}{c}\text{System } S\\ E_1\\ E_2\\ \vdots\\ E_m\end{array}} \underset{E_i \to E_i + \lambda E_j}{\rightsquigarrow} \boxed{\begin{array}{c}\text{System } S'\\ E_i' = E_i + \lambda E_j\\ \text{rest unchanged}\\ \vdots\end{array}} \underset{\substack{E_i' \to\\ E_i' - \lambda E_j}}{\rightsquigarrow} \boxed{\begin{array}{c}\text{Back to the}\\ \text{original system}\\ S\end{array}}$$

$\square$

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* Optional but instructive.

(1.5) Thus in order to solve a linear system, we proceed as follows:

- use elementary operations to simplify the system
- solve the simpler system (we will state later what that means).

e.g.

$$\begin{aligned} E_1 &: \quad x_1 - x_2 + x_3 = 1 \\ E_2 &: \quad x_1 + x_2 - x_3 = 5 \\ E_3 &: \quad x_1 + 2x_2 + 4x_3 = 10 \end{aligned}$$

Combine $\rightsquigarrow$ $E_2 \to E_2 - E_1$, $E_3 \to E_3 - E_1$

$$\boxed{\begin{array}{r} x_1 - x_2 + x_3 = 1 \\ 2x_2 - 2x_3 = 4 \\ 3x_2 + 3x_3 = 9 \end{array}}$$

Scale $\rightsquigarrow$ $E_2 \to \frac{1}{2} E_2$, $E_3 \to \frac{1}{3} E_3$

$$\boxed{\begin{array}{r} x_1 - x_2 + x_3 = 1 \\ x_2 - x_3 = 2 \\ x_2 + x_3 = 3 \end{array}}$$

$E_3 \to E_3 - E_2$ $\rightsquigarrow$ Combine

$$\boxed{\begin{array}{r} x_1 - x_2 + x_3 = 1 \\ x_2 - x_3 = 2 \\ 2x_3 = 1 \end{array}}$$

So $x_3 = \frac{1}{2}$; $x_2 = 2 + x_3 = 2 + \frac{1}{2} = \frac{5}{2}$

$$x_1 = 1 + x_2 - x_3 = 1 + \frac{5}{2} - \frac{1}{2} = 3$$

Check : (original system) $x_1 - x_2 + x_3 = 3 - \frac{5}{2} + \frac{1}{2} = \frac{6-5+1}{2} = 1$

$$x_1 + x_2 - x_3 = 3 + \frac{5}{2} - \frac{1}{2} = \frac{6+5-1}{2} = 5 \quad ✓$$

$$x_1 + 2x_2 + 4x_3 = 3 + 2\left(\frac{5}{2}\right) + 4\left(\frac{1}{2}\right) = 3 + 5 + 2 = 10 \quad ✓$$

(1.6) In the matrix representation of a linear system, the three elementary operations are called (elementary) <u>row operations</u>.

I. (Swap) - Interchange two rows.
II. (Scale) - Multiply a row by a non-zero number.
III. (Combine) - Add a multiple of a row to another row.

Using these operations, we can "simplify" the matrix. More precisely, starting from an arbitrary matrix B, we can use the operations I - III above to bring it to <u>echelon form</u>. [Gauss-Jordan algorithm - next time]

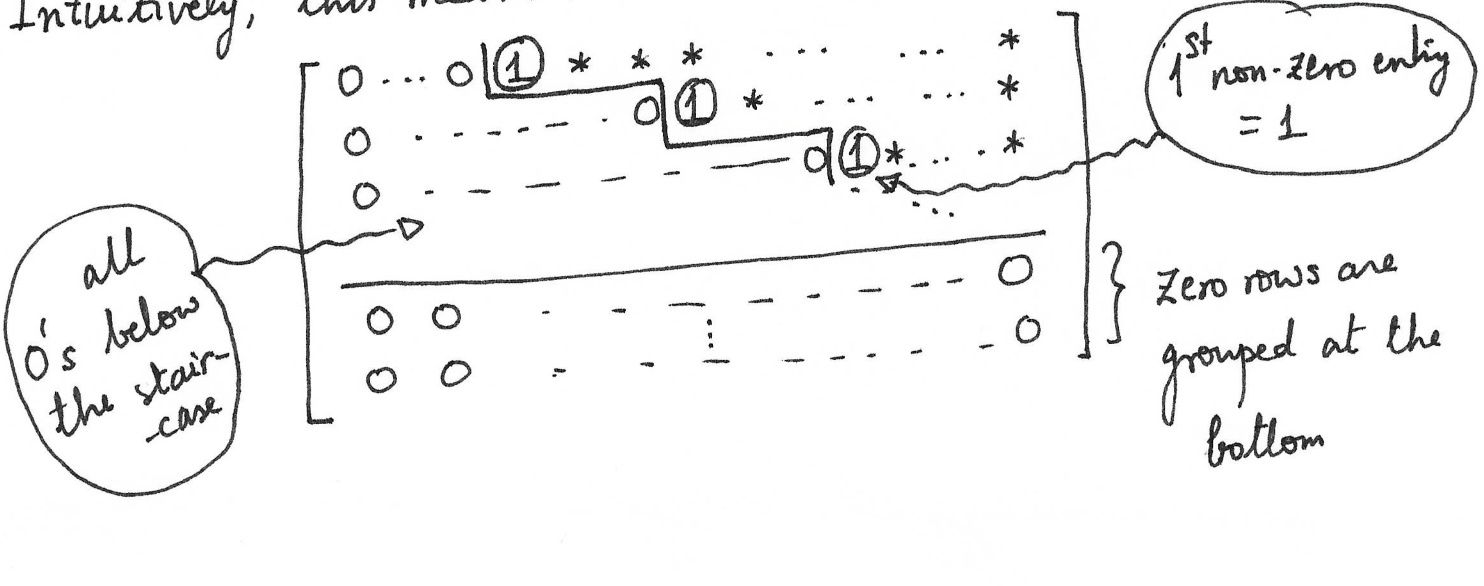
Definition. - An $m \times n$ matrix B is said to be in <u>echelon form</u> if :

(i) all rows containing only zeroes are at the bottom.

(ii) in every non-zero row, the first (from left) non-zero entry is 1.

(iii) if a row is non-zero, its first non-zero entry is to the <u>right</u> of the first non-zero entry of the previous row.

Intuitively, this means that the matrix has a "staircase shape"



e.g. $\begin{bmatrix} 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is in echelon* form

$\begin{bmatrix} 1 & -1 & 0 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is also in echelon form.

$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 0 & 0 & 4 \end{bmatrix}$ NOT in echelon form. It can be brought to an echelon form using the elementary row operations:

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 0 & 0 & 4 \end{bmatrix} \xrightarrow[R_1 \leftrightarrow R_2]{\text{Swap}} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix} \xrightarrow[R_3 \to \frac{1}{4} R_3]{\text{Scale}} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

echelon form ✓

(1.7) If the augmented matrix of a linear system is in echelon form, then the system is (very) easy to solve.

The elimination algorithm (Gauss-Jordan) produces a matrix in an echelon form, starting from any matrix, and using the elementary row operations (Next time).

Let us work out an example which makes this algorithm transparent.

Example. (4 variables & 3 equations)

$$\begin{aligned} x_2 + x_3 - x_4 &= 3 \\ x_1 + 2x_2 - x_3 + x_4 &= 2 \\ -x_1 + x_2 + 7x_3 - x_4 &= 1 \end{aligned}$$

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* echelon - from French word échelle meaning ladder

- Matrix (augmented):

$$B = \left[\begin{array}{cccc|c} 0 & 1 & 1 & -1 & 3 \\ 1 & 2 & -1 & 1 & 2 \\ -1 & 1 & 7 & -1 & 1 \end{array}\right]$$

NOT in echelon form

$$\xrightarrow[R_1 \leftrightarrow R_2]{\text{Swap}} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 2 \\ 0 & 1 & 1 & -1 & 3 \\ -1 & 1 & 7 & -1 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 + R_1]{\text{Combine}} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 2 \\ 0 & 1 & 1 & -1 & 3 \\ 0 & 3 & 6 & 0 & 3 \end{array}\right]$$

$$\xrightarrow[R_3 \to \frac{1}{3} R_3]{\text{Scale}} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 2 \\ 0 & 1 & 1 & -1 & 3 \\ 0 & 1 & 2 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_2]{\text{combine}} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 2 \\ 0 & 1 & 1 & -1 & 3 \\ 0 & 0 & 1 & 1 & -2 \end{array}\right]$$

echelon form ✓

The last matrix represents the system:

$$\begin{aligned} x_1 + 2x_2 - x_3 + x_4 &= 2 \\ x_2 + x_3 - x_4 &= 3 \\ x_3 + x_4 &= -2 \end{aligned}$$

for any choice of $x_4 = t$ (a real number) we get a solution

$$\begin{aligned} x_3 &= -2 - x_4 = -2 - t \\ x_2 &= 3 - x_3 + x_4 = 3 - (-2-t) + t = 5 + 2t \\ x_1 &= 2 - 2x_2 + x_3 - x_4 = 2 - 2(5+2t) + (-2-t) - t \\ &= 2 - 10 - 4t - 2 - 2t = -10 - 6t. \end{aligned}$$

Thus there are infinitely many solutions:

$$\boxed{\begin{aligned} x_1 &= -10 - 6t \\ x_2 &= 5 + 2t \\ x_3 &= -2 - t \\ x_4 &= t \end{aligned}}$$