

(2.0) Recall: last time we learnt how to encode a linear system into a matrix, called the augmented matrix of the system. We listed three elementary operations which do not change the set of solutions of a linear system:

Linear Systems		↔	Matrices	Elementary Operations
Swap	$E_i \leftrightarrow E_j$		$R_i \leftrightarrow R_j$	
Scale	$E_i \rightarrow \alpha E_i$		$R_i \rightarrow \alpha R_i$	
Combine	$E_i \rightarrow E_i + \lambda E_j \quad (j \neq i)$		$R_i \rightarrow R_i + \lambda R_j$	

Here :

- E_1, E_2, \dots, E_m are the linear equations our system is composed of.
- R_1, R_2, \dots, R_m are the rows of an $(m \times n)$ -matrix.
- $\alpha \neq 0$ and λ arbitrary numbers

Strategy to solve a linear system: simplify the system by performing the elementary operations. Solve the simpler system.

By "simpler system" we mean a linear system whose augmented matrix is in echelon form.

Today we will learn that this strategy always works. That is, any matrix can be put in echelon form using the three elementary operations.

(2.1) Echelon and reduced echelon forms. Recall from the

last lecture that a matrix B is in echelon form if it has the following "staircase shape"

\uparrow 1st non-zero entry of each non-zero row is 1

$$\begin{bmatrix} 0 & \dots & 0 & \textcircled{1} & * & * & \dots & * & \dots & * \\ 0 & \dots & \dots & \dots & 0 & \textcircled{1} & * & \dots & \dots & * \\ 0 & \dots & \dots & \dots & \dots & \dots & 0 & \textcircled{1} & * & \dots & * \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \hline 0 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \end{bmatrix}$$

\uparrow 1st non-zero entry of a row is to the right of the one from the previous row

rows consisting entirely of zeroes are grouped at the bottom

We say B is in a reduced echelon form if

- (i) B is in echelon form; and
- (ii) if $b_{ij} = 1$ is the 1st non-zero entry of row R_i then it is the only non-zero entry of the column C_j .

e.g. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is in reduced echelon form.

$\begin{bmatrix} 1 & 5 & 0 & 7 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is also in reduced echelon form.

$B = \begin{bmatrix} 0 & 1 & 4 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is in echelon, but not in reduced echelon form because $b_{15} = 3 \neq 0$

$R_1 \rightarrow R_1 - 3R_2 \rightarrow \begin{bmatrix} 0 & 1 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ reduced \checkmark

(2.2) Theorem. Given any matrix B , we can put B in a reduced echelon form using the three elementary operations.

This is achieved via two protocols. (or algorithms).

I. Input : an $(m \times n)$ -matrix B .
Output : $(m \times n)$ -matrix $E(B)$ in echelon form, obtained from B by a sequence of elementary operations.

Step 0. If B is O (i.e., if all entries of B are 0 's) then return $E(B) = O$.

Step 1. Find smallest i such that $b_{i1} \neq 0$.
• if there is no such b_{i1} , then the 1^{st} column of B consists entirely of 0 's. That is, $B = \begin{bmatrix} 0 & \boxed{B'} \\ 0 & \\ \vdots & \\ 0 & \end{bmatrix}$
 $m \times (n-1)$ matrix

In this case, run the protocol for the (smaller) matrix B' to find $E(B')$ and return

$$E(B) = \begin{bmatrix} 0 & \boxed{E(B')} \\ 0 & \\ \vdots & \\ 0 & \end{bmatrix}$$

Step 2. - Let $\alpha = b_{i1} (\neq 0)$. Swap $R_1 \leftrightarrow R_i$ and
Scale $R_1 \rightarrow \frac{1}{\alpha} R_1$.
(after this step, $(1^{st} \text{ row}, 1^{st} \text{ column})$ -entry = 1)

Step 3. For each $j = 2, \dots, m$; Combine
 $R_j \rightarrow R_j - b_{j1} R_1$.

Step 4. Now our matrix has the following form:

$$B = \begin{bmatrix} \textcircled{1} & * & * & \dots & * \\ 0 & \boxed{B''} & & & \\ 0 & & & & \\ \vdots & & & & \\ 0 & & & & \end{bmatrix}$$

B''
(m-1) x (n-1) size matrix

Run the protocol on the smaller matrix B'' to get $E(B'')$.

Return $E(B) = \begin{bmatrix} 1 & * & * & \dots & * \\ 0 & \boxed{E(B'')} & & & \\ 0 & & & & \\ \vdots & & & & \\ 0 & & & & \end{bmatrix}$.

II. Input : an (m x n) matrix C in an echelon form.
 Output : (m x n) matrix $RE(C)$ in a reduced echelon form,
 obtained from C by elementary operations.

For this, we scan the matrix from bottom to top.

Step 0. If $C = 0$, Return $RE(C) = 0$.
 (If C is empty, return $RE(C) = \text{empty}$.)

Step 1. Find largest k such that k^{th} row is non-zero.
 Let l be such that $C_{k,l} = 1$ is the 1^{st} non-zero entry of $\textcircled{1}$ the k^{th} row of C .

Step 2. For each $i = 1, 2, \dots, k$:
 (Combine) $R_i \rightarrow R_i - C_{i,l} \cdot R_k$

Now our matrix is in the form $G = \begin{bmatrix} \boxed{C'} & 0 & * & \dots & * \\ 0 & \boxed{1} & 0 & * & \dots & * \\ 0 & \dots & \textcircled{1} & * & * & \dots & * \\ \vdots & & & & & & \\ 0 & & & & & & 0 \\ 0 & & & & & & 0 \end{bmatrix}$

Let C' be the top ~~(k-1)~~ $(k-1) \times n$ matrix

k^{th} row

l^{th} column

Step 3. Repeat the protocol on C' to get $RE(C')$ (5)

return $RE(G) = \left[\begin{array}{c|ccc} RE(C') & & & \\ \hline 0 & \dots & 0 & \textcircled{1} * \dots * \\ \hline 0 & \dots & \dots & 0 \\ 0 & \dots & \dots & 0 \end{array} \right] \leftarrow k^{\text{th}} \text{ row of } G$

(2.3) Example. (i) Let $B = \begin{bmatrix} 1 & 3 \\ 4 & 10 \end{bmatrix}$.

$E(B) : \begin{bmatrix} \textcircled{1} & 3 \\ 4 & 10 \end{bmatrix} \xrightarrow{R_2 \rightarrow R_2 - 4R_1} \begin{bmatrix} 1 & 3 \\ 0 & \textcircled{-2} \end{bmatrix} \xrightarrow{R_2 \rightarrow \frac{-1}{2}R_2} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = E(B) \checkmark$

$RE(B) : \begin{bmatrix} 1 & 3 \\ 0 & \textcircled{1} \end{bmatrix} \xrightarrow{R_1 \rightarrow R_1 - 3R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = RE(B) \checkmark$

(ii) Let $B = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 2 & -1 & 7 & 8 \\ -1 & 1 & -5 & -5 \end{bmatrix}$

$E(B) : \begin{bmatrix} \textcircled{1} & 1 & -1 & 1 \\ 2 & -1 & 7 & 8 \\ -1 & 1 & -5 & -5 \end{bmatrix} \xrightarrow{\substack{R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 + R_1}} \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & \textcircled{-3} & 9 & 6 \\ 0 & 2 & -6 & -4 \end{bmatrix}$

$\xrightarrow{R_2 \rightarrow \frac{-1}{3}R_2} \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & \textcircled{1} & -3 & -2 \\ 0 & 2 & -6 & -4 \end{bmatrix} \xrightarrow{R_3 \rightarrow R_3 - 2R_2} \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = E(B) \checkmark$

$RE(B) : \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & \textcircled{1} & -3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \rightarrow R_1 - R_2} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = RE(B) \checkmark$

(2.4) Linear system represented by a matrix in reduced echelon form.

- If it contains the following row - $[0 \ 0 \ \dots \ 0 \ | \ 1]$ then the corresponding linear system is inconsistent (i.e., has no solutions).
- The variables which correspond to columns containing pivoted 1's (corners of our staircase) are called dependent variables.
- The variables that are not dependent are called independent (or free parameters).

e.g. Take the matrix $B = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 2 & -1 & 7 & 8 \\ -1 & 1 & -5 & -5 \end{bmatrix}$ from the last page.

$$RE(B) = \begin{array}{c} \text{1st} \\ \downarrow \\ \begin{array}{c|ccc} \textcircled{1} & 0 & 2 & 3 \\ 0 & \textcircled{1} & -3 & -2 \\ 0 & 0 & 0 & 0 \end{array} \end{array}$$

2nd
↓

(pivoted 1's)

We conclude immediately that

(i) The system
$$\begin{cases} x_1 + x_2 - x_3 = 1 \\ 2x_1 - x_2 + 7x_3 = 8 \\ -x_1 + x_2 - 5x_3 = -5 \end{cases}$$
 is consistent.

(ii) x_3 is a free parameter. Write the system represented by

$$RE(B): \begin{cases} x_1 + 2x_3 = 3 \\ x_2 - 3x_3 = -2 \end{cases}$$

(iii) For every choice of $x_3 = t$

$$\begin{cases} x_1 = 3 - 2t \\ x_2 = -2 + 3t \end{cases} \text{ is a solution.}$$

(2.5) Example. Consider the system

$$\begin{aligned} x_1 + x_2 - x_5 &= 1 \\ x_2 + 2x_3 + x_4 + 3x_5 &= 1 \\ x_1 - x_3 + x_4 + x_5 &= 0 \end{aligned}$$

Determine whether this system is consistent or not. If yes, describe its set of solutions.

Sol: $B = \left[\begin{array}{ccccc|c} \textcircled{1} & 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 2 & 1 & 3 & 1 \\ 1 & 0 & -1 & 1 & 1 & 0 \end{array} \right]$

$\xrightarrow{R_3 \rightarrow R_3 - R_1}$ $\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & -1 & 1 \\ 0 & \textcircled{1} & 2 & 1 & 3 & 1 \\ 0 & -1 & -1 & 1 & 2 & -1 \end{array} \right] \xrightarrow{R_3 \rightarrow R_3 + R_2}$ $\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 2 & 1 & 3 & 1 \\ 0 & 0 & \textcircled{1} & 2 & 5 & 0 \end{array} \right]$

$E(B)$ - echelon form.

$\xrightarrow{R_2 \rightarrow R_2 - 2R_3}$ $\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & -1 & 1 \\ 0 & \textcircled{1} & 0 & -3 & -7 & 1 \\ 0 & 0 & 1 & 2 & 5 & 0 \end{array} \right] \xrightarrow{R_1 \rightarrow R_1 - R_2}$ $\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 3 & 6 & 0 \\ 0 & 1 & 0 & -3 & -7 & 1 \\ 0 & 0 & 1 & 2 & 5 & 0 \end{array} \right]$

$R(B)$ reduced echelon

The system is consistent. ✓

System corresponding to $R(B)$:
 x_4 and x_5 are free parameters.

$$\begin{aligned} x_1 + 3x_4 + 6x_5 &= 0 \\ x_2 - 3x_4 - 7x_5 &= 1 \\ x_3 + 2x_4 + 5x_5 &= 0 \end{aligned}$$

for every choice

$$\begin{cases} x_4 = s \\ x_5 = t \end{cases}$$

of 2 numbers

\Rightarrow we have a sol-n.

$$\begin{aligned} x_1 &= -3s - 6t \\ x_2 &= 1 + 3s + 7t \\ x_3 &= -2s - 5t \end{aligned}$$

(2.6) Some terminology.-

We say two linear systems are equivalent if they have the same set of solutions.

Two matrices B_1 and B_2 are said to be row equivalent if we can go from B_1 to B_2 by performing elementary row operations. Recall (from Lecture 1) that the row operations are reversible. Thus, for example, if $RE(B_1) = RE(B_2)$ then B_1 and B_2 are row equivalent.

$$B_1 \xrightarrow{\text{row operations}} RE(B_1) = RE(B_2) \xleftarrow{\text{row operations}} B_2$$

e.g. let $B_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B_2 = \begin{bmatrix} 1 & 3 \\ -1 & 5 \end{bmatrix}$

$$B_1 = \begin{bmatrix} \textcircled{1} & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{R_2 \rightarrow R_2 - 3R_1} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \xrightarrow{R_2 \rightarrow -\frac{1}{2}R_2} \begin{bmatrix} 1 & 2 \\ 0 & \textcircled{1} \end{bmatrix} \xrightarrow{\substack{R_1 \rightarrow R_1 \\ -2R_2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = RE(B_1)$$

$$B_2 = \begin{bmatrix} 1 & 3 \\ -1 & 5 \end{bmatrix} \xrightarrow{R_2 \rightarrow R_2 + R_1} \begin{bmatrix} 1 & 3 \\ 0 & 8 \end{bmatrix} \xrightarrow{R_2 \rightarrow \frac{1}{8}R_2} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \xrightarrow{\substack{R_1 \rightarrow R_1 \\ -3R_2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = RE(B_2)$$

So B_1 and B_2 are row equivalent.