Lecture 5

(5.0) Recall - last time we started the study of matrices. We introduced three (algebraic) operations :

Operation	Input	Output
• Addition of matrices	Two matrices $A = (a_{ij})$ and $B = (b_{ij})$ of same size $(m \times n)$	$A+B$: matrix of size $m \times n$ given by $(A+B)_{ij} = a_{ij} + b_{ij}$
• Scalar multiplication	λ : a number $A = (a_{ij})$ an $m \times n$ matrix	λA : matrix of size $m \times n$ given by $(\lambda A)_{ij} = \lambda a_{ij}$
• Matrix multiplication	Two matrices $A = (a_{ij})$ $m \times \underline{\underline{n}}$ $B = (b_{kl})$ $\underline{\underline{n}} \times s$ (# Cols of A = # Rows of B)	$A \cdot B$: matrix of size $\underline{\underline{m}} \times \underline{\underline{s}}$ #Rows of A # Cols. of B given by :

$$\boxed{\begin{array}{l} (A \cdot B)_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \ldots + a_{in} b_{nj} \\ (\text{for every } 1 \leq i \leq m \text{ and } 1 \leq j \leq s) \end{array}}$$

Today's lecture is focused solely on matrix multiplication, to help us get used to the definition.

(5.1) Vectors in n-dimensional space.

Let n be a positive integer (so, $n = 1, 2, 3, \ldots$)

$\mathbb{R}$ = set of all real numbers.

$\mathbb{R}^n$ consists of ~~an~~ ordered n-tuples of real numbers.

Its elements are called n-dimensional vectors and are written as $n \times 1$ matrices (or 1 column of n numbers):

e.g. $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ is a 3-dimensional vector (same as a 3×1 matrix).

$\vec{w} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 3 \\ 4 \end{bmatrix}$ is a 5-dimensional vector (5×1 matrix).

To make a distinction from ordinary variables or numbers, I will indicate a vector by drawing an arrow on its head.

The book uses boldface font for this.

Thus $$\mathbb{R}^n = \left\{ \vec{v} \ : \ \vec{v} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ where } x_1, x_2, \ldots, x_n \text{ are real numbers} \right\}$$

(5.2) Scalar product or dot product.

Given two vectors (n-dimensional) $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

the dot product (or scalar product) is defined as

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$

$$= \sum_{i=1}^{n} u_i v_i \quad \text{(in summation/sigma notation).}$$

This is analogous to how dot product was defined in Calculus, for vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$. We will discuss it again in more detail later in the course.

Thus, another way to think of matrix multiplication is :

$(A \cdot B)_{ij}$ = dot product of i^{th} row of A and j^{th} column of B (both must be vectors of same dimension i.e. # Cols of A must be equal to # Rows of B.)

(5.3) Writing a linear system as an equation for matrices.

Assume we have a linear system of equations :

$$(*)\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 & -\ (\text{eq } 1) \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 & -\ (\text{eq } 2) \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m & -\ (\text{eq } m) \end{cases}$$

Let $A = (a_{ij})_{\substack{1\le i\le m \\ 1\le j\le n}}$ be the $m\times n$ coefficient matrix

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

n-dimensional vector $\qquad$ m-dimensional vector

Then $A\cdot\vec{x}$ is defined ($\underset{m\times n}{A}$ $\underset{n\times 1}{\vec{x}}$ $\leadsto A\cdot\vec{x}$ is $m\times 1$ matrix).

it is of size $m\times 1$, i.e. an m-dimensional vector, given by:

$$A\cdot\vec{x} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & & \vdots & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}}_{m\times n} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{n\times 1} = \underbrace{\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots a_{mn}x_n \end{bmatrix}}_{m\times 1}$$

Hence, $A \cdot \vec{x}$ is an m-dimensional vector, whose entries are the expressions on the left-hand side of the linear system (*).

The linear system of equations (*) <u>can be equivalently written</u> as 1 equation for 2 m-dimensional vectors

$$\boxed{A \cdot \vec{x} = \vec{b}} \quad \text{(both sides are in } \mathbb{R}^m\text{)}.$$

where A ($m \times n$ matrix) and $\vec{b}$ (m-dimensional vector) are given, and we want to find $\vec{x}$ (n-dimensional vector).

(5.4) <u>Solutions of a linear system of equations as vectors</u>

Writing a linear system of equations as $\boxed{A\vec{x} = \vec{b}}$ suggests that its solutions should also be written as (n-dimensional) vectors. For instance:

e.g. Assume $B = \left[\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]$ be the augmented matrix of the linear system.

$$\boxed{\begin{aligned} x_1 - 2x_2 &= 3 \\ x_3 &= 2 \\ x_4 &= 4 \end{aligned}}$$

For this, x_2 is a free/independent variable; and a solution is of the form

$$\begin{aligned} x_1 &= 3 + 2t \\ x_2 &= t \\ x_3 &= 2 \\ x_4 &= 4 \end{aligned}$$

It is often written (as vectors):

$$\vec{x} = \begin{bmatrix} 3 \\ 0 \\ 2 \\ 4 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad \text{where } t \in \mathbb{R}.$$

a particular solution of the given system

$$\boxed{\begin{aligned} x_1 - 2x_2 &= 3 \\ x_3 &= 2 \\ x_4 &= 4 \end{aligned}}$$

a particular solution of the homogeneous system

$$\boxed{\begin{aligned} x_1 - 2x_2 &= 0 \\ x_3 &= 0 \\ x_4 &= 0 \end{aligned}}$$

(5.5) Matrix multiplication as "composing linear functions."

Assume we have 3 variables x_1, x_2, x_3; and we set

$$\begin{aligned} y_1 &= x_1 - x_2 + x_3 \\ y_2 &= x_2 + x_3 \end{aligned}$$

As we saw in (5.3) above, we can rewrite this as:

$$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A \cdot \vec{x} \quad \text{where}$$

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}_{2\times 3}.$$

If we further put, $\begin{aligned} z_1 &= 2y_1 + 3y_2 \\ z_2 &= y_1 - 4y_2 \end{aligned}$ and want to express z_1, z_2 in terms of x_1, x_2, x_3

we end up multiplying matrices, as follows.

$$\vec{y} = A\vec{x} \quad ; \quad \vec{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix} \underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{A\vec{x}}$$

$$= \underbrace{\begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 5 \\ 1 & -5 & -3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 2x_1 + x_2 + 5x_3 \\ x_1 - 5x_2 - 3x_3 \end{bmatrix}.$$

So $z_1 = 2x_1 + x_2 + 5x_3$

$z_2 = x_1 - 5x_2 - 3x_3$.