(8.0) Recall - last time we defined invertible matrices.

An $n \times n$ matrix A is invertible if there exists another $n \times n$ matrix B such that $\boxed{AB = I_n = BA}$ - (*)

Here I_n is the identity matrix of size $n \times n$.

$$I_n = \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ 0 & 0 & \cdots & \vdots \\ \vdots & \vdots & & 0 \\ 0 & 0 & & 1 \end{bmatrix}.$$ Its columns are denoted by

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad \vec{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^n.$$

- We proved that if B as in (*) exists, then it is unique, called the inverse of A, denoted by A^{-1}.
- We proved that if A is invertible, then every linear system $A\vec{x} = \vec{b}$ has a unique solution.

In other words, the reduced echelon form of A is I_n

- If we solve $\boxed{\begin{matrix} A\vec{x}_1 = \vec{e}_1 \\ \vdots \\ A\vec{x}_n = \vec{e}_n \end{matrix}}$ and collect the solutions

n linear systems

$$X = [\vec{x}_1 \; \vec{x}_2 \; \cdots \; \vec{x}_n]$$

Then: $\boxed{AX = I_n}$.

This gave us a method to find the inverse of A.

Input	Procedure	Output
A: $n \times n$ matrix	Gauss-Jordan method applied to $[A \mid I_n] \dashrightarrow [I_n \mid B]$	$\boxed{B = A^{-1}}$

$[I_n \mid B]$: reduced echelon form of A

(8.1) Technicality. - Strictly speaking, what we proved last time was that if $[A \mid I_n] \xrightarrow[\text{Gauss-Jordan}]{} [I_n \mid B]$ then $\boxed{AB = I_n}$. We still have to prove that $BA = I_n$. To see this, we first note that B is also row-equivalent to I_n - since all row operations are reversible. Simply apply the ones from $[A \mid I_n] \dashrightarrow [I_n \mid B]$ in the reverse order. If we follow the procedure written above to $[B \mid I_n] \dashrightarrow [I_n \mid C]$ we have $BC = I_n$. (replace A by B and B by C in the 1^{st} line of this paragraph).

$$\boxed{AB = I_n \text{ and } BC = I_n \text{ implies } A = C}$$

Proof. $\boxed{A} = A \cdot I_n = A \cdot (BC) = (AB)C = I_n \cdot C = \boxed{C}$.

associativity of matrix mult.

□ (end of the proof).

(8.2) Example. Let $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{bmatrix}$. Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -2 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_1 \leftrightarrow R_2]{} \left[\begin{array}{ccc|ccc} -1 & 2 & -2 & 0 & 1 & 0 \\ 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \to (-1)R_1} \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 0 & -1 & 0 \\ 0 & 3 & -4 & 1 & 2 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 0 & -1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \\ 0 & 3 & -4 & 1 & 2 & 0 \end{array}\right] \xrightarrow{R_2 \to (-1)R_2} \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 & 0 & -1 \\ 0 & 3 & -4 & 1 & 2 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - 3R_2} \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 & 0 & -1 \\ 0 & 0 & 2 & 1 & 2 & 3 \end{array}\right] \xrightarrow[R_3 \to \frac{1}{2}R_3]{} \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 & 0 & -1 \\ 0 & 0 & 1 & \frac{1}{2} & 1 & \frac{3}{2} \end{array}\right] \xrightarrow{\substack{R_1 \to R_1 - 2R_3 \\ R_2 \to R_2 + 2R_3}} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & -1 & -3 & -3 \\ 0 & 1 & 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & \frac{1}{2} & 1 & \frac{3}{2} \end{array}\right]$$

$$\xrightarrow[R_1 \to R_1 + 2R_2]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & \frac{1}{2} & 1 & \frac{3}{2} \end{array}\right].$$

Claim $A^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ \frac{1}{2} & 1 & \frac{3}{2} \end{bmatrix}$

Check
$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ \frac{1}{2} & 1 & \frac{3}{2} \end{bmatrix} = \begin{bmatrix} 2-1+0 & 2-2+0 & 2-2+0 \\ -1+2-1 & -1+4-2 & -1+4-3 \\ 0-1+1 & 0-2+2 & 0-2+3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

We don't have to check $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ \frac{1}{2} & 1 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. It follows automatically.

(8·3) Linear dependence and independence. (m, n : positive integers)

Assume we are given a set of vectors in $\mathbb{R}^m$, say

$$\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} \quad (n \text{ vectors in } \mathbb{R}^m).$$

We say that this set of vectors is linearly dependent if we can find n numbers, $x_1, x_2, \ldots, x_n \in \mathbb{R}$ NOT ALL zero such that

$$\boxed{x_1\vec{v}_1 + x_2\vec{v}_2 + \ldots + x_n\vec{v}_n = \vec{0}} \leftarrow \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

Zero vector in $\mathbb{R}^m$.

Otherwise we say that $\vec{v}_1, \ldots, \vec{v}_n$ are linearly independent.

Meaning if the only way to make $\boxed{x_1\vec{v}_1 + x_2\vec{v}_2 + \ldots + x_n\vec{v}_n} = \vec{0}$ is to set $x_1 = 0, x_2 = 0, \ldots, x_n = 0$; then $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a set of linearly independent vectors

This is just a new terminology. The expression

$$\boxed{x_1\vec{v}_1 + x_2\vec{v} + \ldots + x_n\vec{v}_n = \vec{0}}$$

same m as in $\mathbb{R}^m$ above

is nothing but a homogeneous system of m equations in n variables.

Linear independence $\longleftrightarrow$ no non-trivial solutions

Linear dependence $\longleftrightarrow$ (yes) non-trivial solutions exist.

Example. Determine whether the following vectors in $\mathbb{R}^3$ are linearly independent or dependent.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}.$$

Solution. $x_1 \vec{v}_1 + x_2 \vec{v}_2 + x_3 \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is same as:

$$\left.\begin{aligned} \boxed{1}x_1 + \boxed{3}x_2 + \boxed{5}x_3 &= 0 \\ \boxed{2}x_1 + \boxed{4}x_2 + \boxed{6}x_3 &= 0 \\ \boxed{-1}x_1 + \boxed{3}x_2 + \boxed{7}x_3 &= 0 \end{aligned}\right\}$$

$\uparrow \vec{v}_1 \quad \uparrow \vec{v}_2 \quad \uparrow \vec{v}_3$

$\leftarrow$ homogeneous linear system of (i.e, R.H.S. all 0's) 3 variables 3 equations.

- Put $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ -1 & 3 & 7 \end{bmatrix}$ in reduced echelon form.

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ -1 & 3 & 7 \end{bmatrix} \xrightarrow[\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + R_1}]{} \begin{bmatrix} 1 & 3 & 5 \\ 0 & -2 & -4 \\ 0 & 6 & 12 \end{bmatrix} \xrightarrow[\substack{R_2 \to \frac{-1}{2}R_2 \\ R_3 \to \frac{1}{6}R_3}]{} \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

echelon ✓

$$\xrightarrow{R_1 \to R_1 - 3R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

reduced echelon ✓

- Solutions to the homogeneous system: $x_1 = t$, $x_2 = -2t$, $x_3 = t \in \mathbb{R}$ arbitrary.

Non-trivial (i.e., not all zero) solutions exist $\Rightarrow$ $\boxed{\text{LINEARLY DEPENDENT}}$

A solution (e.g. put $t=1$ to get $x_1=1$, $x_2=-2$, $x_3=1$) gives a <u>linear relation</u> among $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$:

$$\boxed{\vec{v}_1 - 2\vec{v}_2 + \vec{v}_3 = \vec{0}}.$$

(This "explains" the term <u>linearly dependent</u>, for example, $\vec{v}_1 = 2\vec{v}_2 - \vec{v}_3$. So $\vec{v}_1$ <u>depends</u> (linearly) on $\vec{v}_2$ and $\vec{v}_3$).

(8.4) Theorem. - If we are given n vectors in $\mathbb{R}^m$, and $n > m$ then these vectors are linearly dependent.

This is a restatement of "if a system has more variables than equations, then it cannot have a unique solution" - see Lecture 3 page 5.

Given n vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ in $\mathbb{R}^m$, the equation

$x_1\vec{v}_1 + x_2\vec{v}_2 + \dots + x_n\vec{v}_n = \vec{0} \leftrightarrow$ homogeneous linear system of m equations in n variables

Since $m < n$, it has either 0 or ∞ly many solutions.
Since it is homogeneous, # Solns $\neq 0$ (recall: homogeneous systems are always consistent - see Lecture 3 page 6.)
Hence there are non-trivial solutions to $x_1\vec{v}_1 + \dots + x_n\vec{v}_n = \vec{0}$ implying that $\vec{v}_1, \dots, \vec{v}_n$ are <u>linearly dependent</u>.

(8.5) Definition. – An $n \times n$ matrix $A = (a_{ij})$ is said to be <u>non-singular</u> if $\vec{x} = \vec{0}$ is the <u>only solution</u> of the homogeneous system $\boxed{A\vec{x} = \vec{0}}$.

On the other hand, if $\boxed{A\vec{x} = \vec{0}}$ admits <u>non-trivial solutions</u> <u>then we say A is singular.</u>

If we write $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ and collect the columns

$\vec{v}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix}, \ldots, \vec{v}_n = \begin{bmatrix} a_{1n} \\ \vdots \\ a_{nn} \end{bmatrix}$: n vectors in $\mathbb{R}^n$, then:

$$\boxed{A\vec{x} = \vec{0}} \xleftrightarrow{\text{same as}} \boxed{x_1\vec{v}_1 + x_2\vec{v}_2 + \ldots + x_n\vec{v}_n = \vec{0}}.$$

Hence, $\boxed{A \text{ is non-singular}} \leftrightarrow \boxed{\text{Its columns are linearly independent}}$

A is singular $\leftrightarrow$ Its columns are linearly dependent.

Example. – If A is invertible, then it is non-singular.

This is because, for invertible A, every $A\vec{x} = \vec{b}$ has a unique solution. Take $\vec{b} = \vec{0}$. We conclude that $\vec{x} = \vec{0}$ is the only solution of $A\vec{x} = \vec{0}$.

The converse of the statement written above is also true. That is, if $\underset{(n\times n)}{A = (a_{ij})}$ is nonsingular, then it is invertible.

Reason. – Let $G = RE(A)$. Since $A\vec{x} = \vec{0}$ has only one solution, $\boxed{\begin{array}{l}\#\text{ non-zero rows of } G = n \\ \text{(also called rank of } A)\end{array}}$, that is $RE(A) = I_n$. So its inverse is given via Gauss-Jordan method.

see Lecture 4 Page 1 for a quick review

(8.6) Summary. – The following statements mean one and the same thing for $A = (a_{ij})$ $n\times n$ matrix.

(1) A is invertible.

(2) Rank of $A = n$.

(3) Reduced echelon form of $A = I_n$.

(4) A is nonsingular

(5) Columns of A are linearly independent.