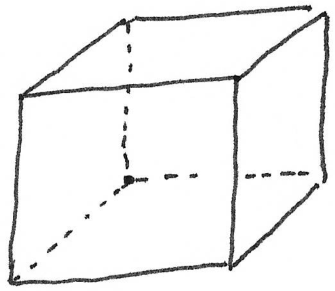
Lecture 12

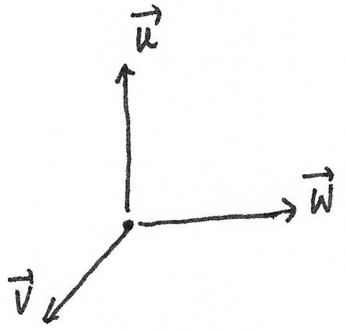
(12.0) Recall - we studied the following operations, from geometric perspective, on the vector spaces $\mathbb{R}^2$ and $\mathbb{R}^3$

- Addition and scalar multiplication (generalizes to $\mathbb{R}^n$).
- Dot product (generalizes to $\mathbb{R}^n$)
- Cross product (ONLY for $\mathbb{R}^3$).

Last time we saw an application of dot and cross products in computing volume of a parallelopipe P.



P = parallelopipe in $\mathbb{R}^3$.

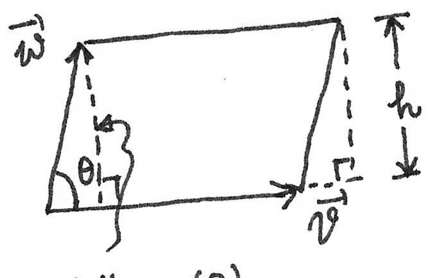


Vectors forming P

$$\boxed{\begin{array}{l}\text{Volume of } P \\ = |\vec{u} \cdot (\vec{v} \times \vec{w})|\end{array}}$$

Let us quickly see why this is true. Our argument is based on the cosine formula for the dot product and sine formula for the cross product.

- Area of the base of P = Area of parallelogram formed by $\vec{v}$ & $\vec{w}$

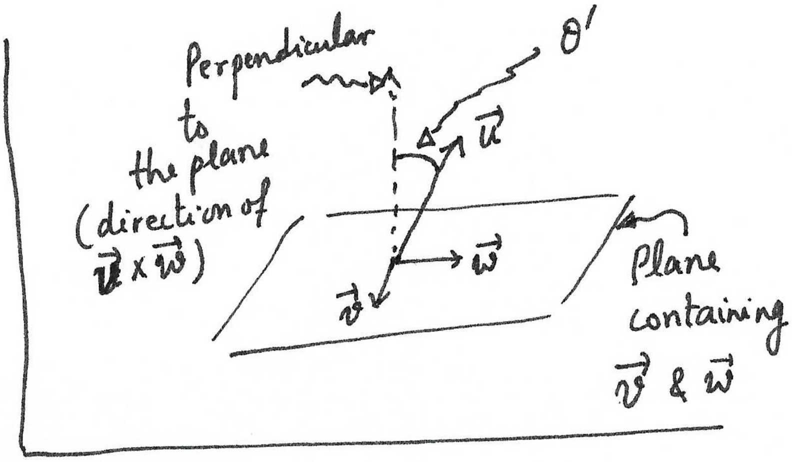


$h = \|\vec{w}\| \sin(\theta)$

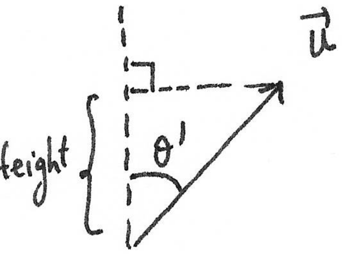
$$= \|\vec{v}\| \cdot h = \|\vec{v}\| \, \|\vec{w}\| \sin(\theta)$$
$$= \|\vec{v} \times \vec{w}\|$$

- Volume of the parallelopipe = (area of the base) $\times$ (height)

Let θ' = angle between $\vec{u}$ and $\vec{v}\times\vec{w}$



Height = $\|\vec{u}\|\,|\cos(\theta')|$



- So Volume = $\underbrace{\|\vec{u}\|\,|\cos(\theta')|}_{\text{Height}}\ \underbrace{\|\vec{v}\times\vec{w}\|}_{\text{Area of the base}}$

$$= |\vec{u}\cdot(\vec{v}\times\vec{w})|$$

$\square$ (end of proof).

Today - we are going to discuss lines (in $\mathbb{R}^2$ and $\mathbb{R}^3$) and planes (in $\mathbb{R}^3$).

(12.1) Lines in $\mathbb{R}^2$. Recall the familiar equation of a line in $\mathbb{R}^2$

$\ell : \boxed{ax+by+c=0}$ $\quad a, b, c \in \mathbb{R}$ (a and b are not both 0)

Usually we split it into 2 cases :

$\underline{b=0}$

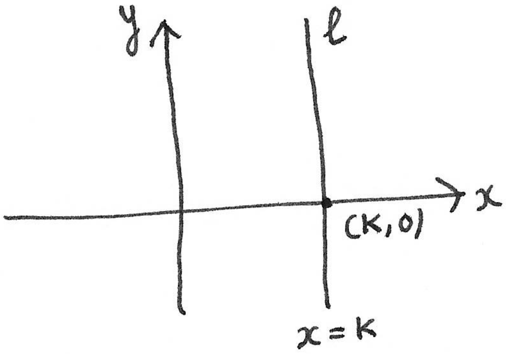
ℓ is a vertical line

$$\boxed{x = -\frac{c}{a}}$$

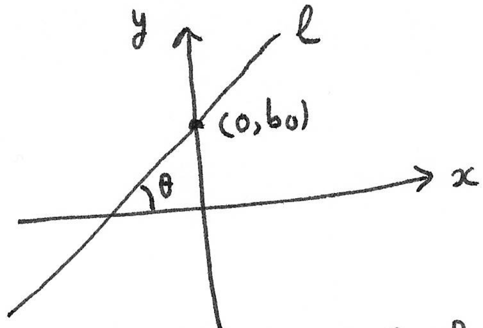
$\underline{b\neq 0}$

$$y = \underbrace{\left(-\frac{a}{b}\right)}_{\text{slope of the line}} x + \underbrace{\left(\frac{-c}{b}\right)}_{y\text{-intercept}}$$

("$y = mx + b_0$ form")



Vertical lines
(all parallel to y-axis)



Not vertical: of the form

$$y = mx + b_0$$

$m = \tan(\theta) =$ slope of ℓ

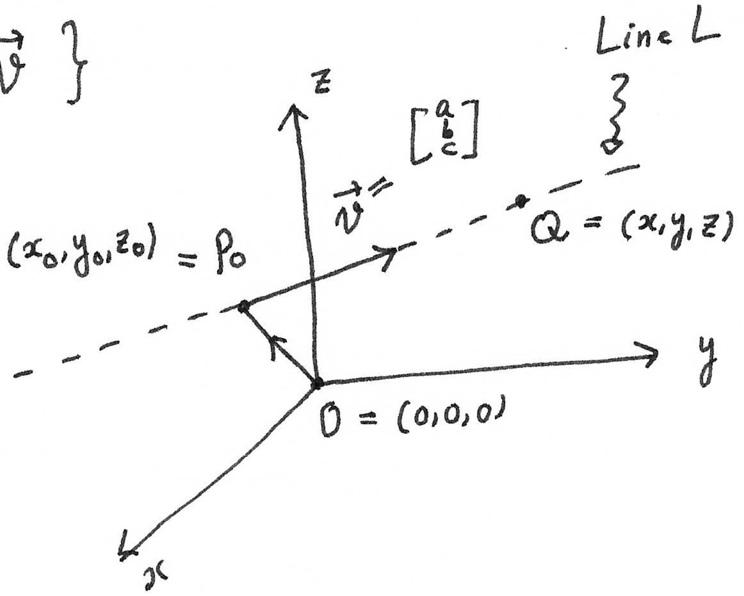
$b_0 =$ y-intercept.

(12.2) Lines in $\mathbb{R}^3$.

A line L in $\mathbb{R}^3$ is given by a point $\underbrace{P_0 = (x_0, y_0, z_0)}_{\text{"INITIAL POINT"}} \in \mathbb{R}^3$

and a vector $\boxed{\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}}$. ← DIRECTION VECTOR

- $L = \{$ all points Q in $\mathbb{R}^3$ such that $\overrightarrow{P_0Q}$ is parallel to $\vec{v}$ $\}$



Alternatively

L consists of all points (x, y, z) such that

$$\begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} = t \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad (t \in \mathbb{R})$$

This allows us to write L in a "parametric form"

$$\begin{aligned} x &= x_0 + a \cdot t \\ y &= y_0 + b \cdot t \\ z &= z_0 + c \cdot t \end{aligned}$$

INITIAL POINT (at $t=0$)

DIRECTION VECTOR

or "vector form" :

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Notice : similarity between equations of lines in $\mathbb{R}^3$, and vector form of solutions of a ~~homogeneous~~ linear system (3 variables, 2 equations - so there is 1 free parameter).

There is another way to write L, by eliminating t from the equations above

(if $a \neq 0$ & $b \neq 0$ & $c \neq 0$) :

$$\boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}}$$

LINEAR SYSTEM
3 variables
2 equations.

Sometimes in the literature called "symmetric form of equations of a line in $\mathbb{R}^3$". This form realizes L as intersection of 2 planes

(12.3) Planes in $\mathbb{R}^3$.

A typical equation of a plane in $\mathbb{R}^3$ is

$$\boxed{ax + by + cz + d = 0}$$

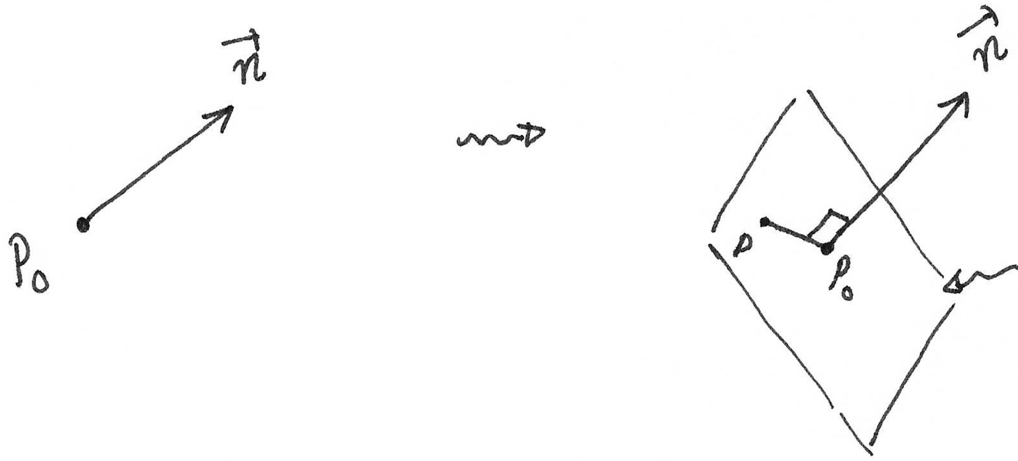
(1 linear equation in 3 variables)

Notice it is analogous to line in $\mathbb{R}^2$ $\boxed{ax+by+c=0}$.

Geometrically, a plane Π in $\mathbb{R}^3$ is given by the data of

- a point P_0 on the plane
- a vector $\vec{n}$ (<u>normal vector</u>)

The plane is then defined as the set of points P of $\mathbb{R}^3$ such that $\overrightarrow{P_0P}$ is orthogonal to $\vec{n}$.



Plane consisting of points P such that $\overrightarrow{P_0P} \perp \vec{n}$

If $\boxed{\begin{array}{l} P_0 = (x_0, y_0, z_0) \\ \vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \\ \text{(given)} \end{array}}$ $P = (x, y, z)$ any point in $\mathbb{R}^3$,

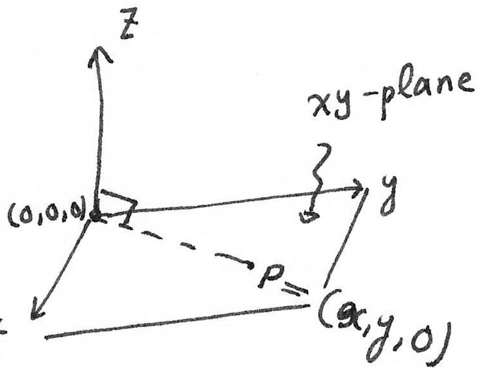
then

$\underline{\overrightarrow{P_0P} \perp \vec{n}}$ $\longleftrightarrow$ $\boxed{\overrightarrow{P_0P} \cdot \vec{n} = 0}$

which gives $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} = 0$

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$$

Examples. (i) xy plane is given by the equation $z = 0$.
It is the plane passing through $(0,0,0)$, perpendicular to $\vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$



(ii) $\boxed{2x + 3y - z = 0}$

$P_0 = (0,0,0)$, $\vec{n} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$.

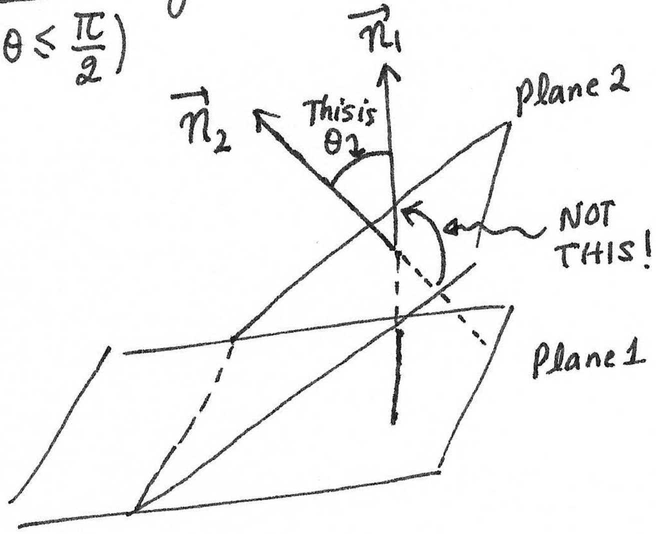
(iii) Find equation of ~~a~~ the plane passing through $(1, -1, 0)$ and perpendicular to $\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$.

Ans. $0(x-1) + 2(y - (-1)) + 3(z - 0) = 0$

$2(y+1) + 3z = 0$

$$\boxed{2y + 3z + 2 = 0}.$$

(12.4) Two planes are said to be perpendicular (or orthogonal) to each other if their normal vectors are orthogonal. In general angle between two planes is the acute angle $(0 \leq \theta \leq \frac{\pi}{2})$ between their normal vectors:



eg.* $x - y + z = 7$ - plane 1

$x + 4y + z = 2$ - plane 2

$\vec{n}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ $\vec{n}_2 = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$

$\vec{n}_1 \cdot \vec{n}_2 = 1 - 4 + 1 = -2$ - so the angle between $\vec{n}_1$ and $\vec{n}_2$ will be obtuse. We can fix that by writing plane 2 as $\boxed{-x - 4y - z = -2}$ and replacing $\vec{n}_2$ by $\begin{bmatrix} -1 \\ -4 \\ -1 \end{bmatrix}$.

$$\cos(\theta) = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{\|\vec{n}_1\| \, \|\vec{n}_2\|} = \frac{2}{\sqrt{3}\sqrt{18}} \qquad \theta = \cos^{-1}\left(\frac{2}{\sqrt{54}}\right).$$

(12.5) Finding the normal vector using cross product:

Find equation of plane that contains $P = (1, 0, 1)$

$Q = (2, 1, 1)$

$R = (3, 1, 2)$

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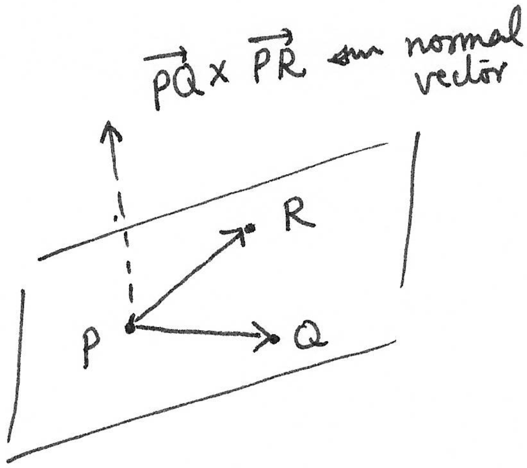
*Optional



Soln.

$$\overrightarrow{PQ} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\overrightarrow{PR} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \det \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} = 1 \cdot \vec{e}_1 - 1 \cdot \vec{e}_2 + (-1)\vec{e}_3$$

$$= \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \vec{n}.$$

So, equation of the plane, containing $P = (1, 0, 1)$ perpendicular to $\vec{n} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ is

$$1(x-1) + (-1)(y-0) + (-1)(z-1) = 0$$

$$x - 1 - y - z + 1 = 0$$

$$\boxed{x - y - z = 0}$$