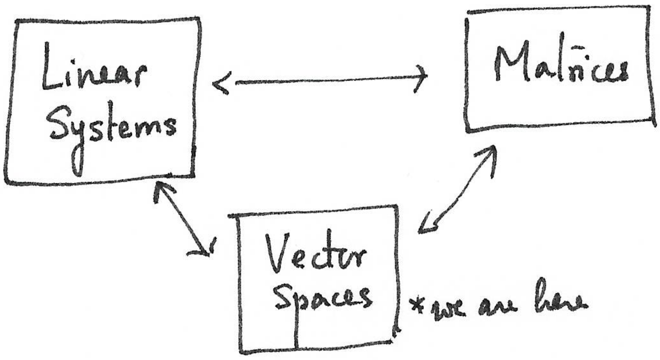
Lecture 13

(13.0) Recall the original 3 components of this course



Last week we revised some of the essential geometric constructions in 2 and 3 dimensional vector spaces – as a warm up for the algebra of vectors in n-dimensions, which is the main topic of our next segment.

(13.1) <u>Vector space $\mathbb{R}^n$</u>. Let $n \geq 1$ be a positive integer. Recall that an element of $\mathbb{R}^n$, called n-dimesional vector, is just an $(n \times 1)$ matrix

$$\mathbb{R}^n = \left\{ \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \text{ where } v_1, v_2, \ldots, v_n \text{ are real numbers} \right\}$$

As with matrices of arbitrary size, we have two main algebraic operations: <u>addition</u> and <u>scalar multiplication</u>

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \quad \rightsquigarrow \quad \vec{u} + \vec{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix} \quad \text{(addition)}$$

$$a \in \mathbb{R}, \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \quad \rightsquigarrow \quad a\vec{v} = \begin{bmatrix} av_1 \\ av_2 \\ \vdots \\ av_n \end{bmatrix} \quad \text{(scalar multiplication)}$$

Let us list the algebraic properties of these operations, as we did earlier in Lecture 6. In that list, an important character was <u>the zero matrix</u>, which as a vector in $\mathbb{R}^n$ is denoted as

$$\text{Zero vector} \quad \vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times 1} \in \mathbb{R}^n .$$

<u>Properties.</u> (Closure properties):

(C1) For $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$, <u>$\vec{u}+\vec{v}$ is in $\mathbb{R}^n$</u>

(C2) For $a \in \mathbb{R}$ and $\vec{v}$ in $\mathbb{R}^n$, <u>$a\vec{v}$ is in $\mathbb{R}^n$</u>

(Properties of addition): $\vec{u}, \vec{v}, \vec{w}$ vectors in $\mathbb{R}^n$.

(A1) $\vec{u}+\vec{v} = \vec{v}+\vec{u}$ (addition is commutative)

(A2) $\vec{u}+(\vec{v}+\vec{w}) = (\vec{u}+\vec{v})+\vec{w}$ (~~ass~~ associative)

(A3) $\vec{0}+\vec{v} = \vec{v}$

(A4) For every $\vec{u}$ in $\mathbb{R}^n$, we have $(-\vec{u})$ so that

$$\vec{u}+(-\vec{u}) = \vec{0}$$

(Properties of multiplication by scalars): $a, b \in \mathbb{R}$; $\vec{u}, \vec{v}$ vectors in $\mathbb{R}^n$.

(M1) $a(b\vec{v}) = (ab)\vec{v}$ (associative)

(M2) $a(\vec{u}+\vec{v}) = a\vec{u}+a\vec{v}$ (distributive over addition)

(M3) $(a+b)\vec{u} = a\vec{u}+b\vec{u}$

(M4) $1\cdot\vec{u} = \vec{u}$

(13.2) Remarks. - Later in the course, these properties will be part of <u>the definition</u> of <u>an abstract vector space</u>.

(13.3) <u>Subspaces of $\mathbb{R}^n$</u>. A subset $V \subset \mathbb{R}^n$ is called a <u>subspace</u> if

(i) $\vec{0}$ is in V.

Closure under addition and scalar multiplication $\rightarrow$

(ii) for any $\vec{u}$ and $\vec{v}$ in V, $\vec{u}+\vec{v}$ is in V.

(iii) for any scalar a and a vector $\vec{v}$ in V, $a\vec{v}$ is in V.

Examples. (i) $V = \{\vec{0}\}$ (set containing only one element, namely zero vector). is a subspace of $\mathbb{R}^n$.

(ii) $V = \mathbb{R}^n$ is clearly a subspace of $\mathbb{R}^n$.

(iii) Subspaces of $\mathbb{R}^2$ are

- $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$
- Lines through the origin
- $\mathbb{R}^2$.

(iv) Subspaces of $\mathbb{R}^3$ are

- $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$
- Lines through the origin
- Planes containing the origin
- $\mathbb{R}^3$

(13.4) Non-examples.

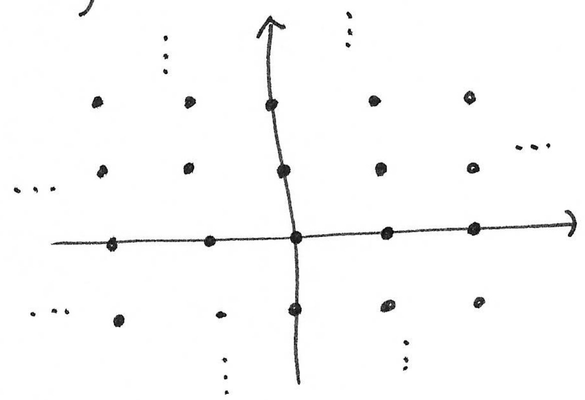
(i) $V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : 2x + y - z = 4 \right\} \subset \mathbb{R}^3$ is <u>NOT</u> a subspace because $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \notin V$.

(ii) $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \text{ and } y \text{ are integers} \right\} \subset \mathbb{R}^2$

<u>NOT</u> a subspace.

- $\vec{0} \in V$ ✓
- $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ in V implies

  $\begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$ in V ✓



(V consists of points with integer coordinates)

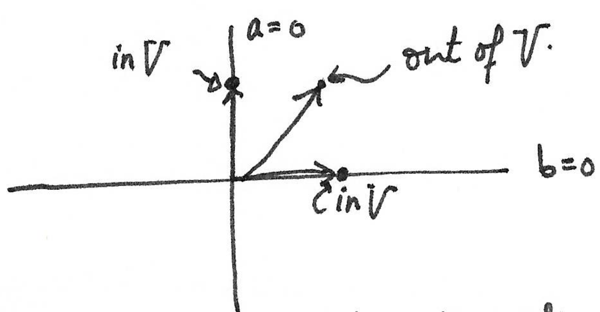
- scalar multiplication ✗

  ($\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in V, $\frac{1}{2} \in \mathbb{R}$ $\rightsquigarrow$ $\frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ <u>NOT</u> in V).

- Union of 2 lines : $V = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} : \begin{matrix} \text{either } a = 0 \\ \text{or } b = 0 \end{matrix} \right\} \subset \mathbb{R}^2$

<u>NOT</u> a subspace :

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ in V but their sum $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not in V



(V is the set of points on these 2 lines)

(13.5) Meta Example I. Solutions to a homogeneous system of linear equations.

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ be an $(m \times n)$ matrix.

Let $V \subset \mathbb{R}^n$ consist of all solutions of $A\vec{x} = \underset{(m\times 1)}{\vec{0}}$

Then <u>V is a subspace of $\mathbb{R}^n$</u>.

- <u>$\underset{(n\times 1)}{\vec{0}} \in V$</u>. This is true because $x_1 = 0, x_2 = 0, \ldots, x_n = 0$ is always a solution to the homogeneous system

  i.e. $\cancel{A}$. $A \cdot \underset{(n\times 1)}{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}} = \underset{(m\times 1)}{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}$

- <u>$\vec{u}, \vec{v}$ in V implies $\vec{u} + \vec{v}$ in V.</u>

$\vec{u}$ and $\vec{v}$ are in V means $A\vec{u} = \vec{0}$ and $A\vec{v} = \vec{0}$.

This gives $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v} = \vec{0} + \vec{0} = \vec{0}$.

So $\vec{u} + \vec{v}$ is in V.

- a is a scalar and $\vec{v}$ a vector in V implies $a\vec{v} \in V$.

$\vec{v}$ in V means $A\vec{v} = 0$. This implies

$$A(a\vec{v}) = a(A\vec{v}) = a \cdot \vec{0} = \vec{0}.$$

(a is a scalar)

e.g. Let $V \subset \mathbb{R}^4$ consist of $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ such that

$$2x_1 + x_2 - x_3 = 0$$

and $$4x_1 + 3x_2 + x_4 = 0.$$

Then V is a subspace.

$$V = \left\{ \vec{x} \text{ in } \mathbb{R}^4 \text{ such that } \begin{bmatrix} 2 & 1 & -1 & 0 \\ 4 & 3 & 0 & 1 \end{bmatrix} \cdot \vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

The Gauss-Jordan method gives us an alternative way to describe this set:

$$\begin{bmatrix} 2 & 1 & -1 & 0 \\ 4 & 3 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 2 & 1 & -1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix} \xrightarrow{R_1 \to R_1 - R_2} \begin{bmatrix} 2 & 0 & -3 & -1 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & \frac{-3}{2} & \frac{-1}{2} \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

free parameters

$$x_1 = \frac{3}{2}x_3 + \frac{1}{2}x_4$$

$$x_2 = -2x_3 - x_4$$

$$V = \left\{ x_3 \begin{bmatrix} \frac{3}{2} \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} \frac{1}{2} \\ -1 \\ 0 \\ 1 \end{bmatrix} \text{ where } x_3, x_4 \in \mathbb{R} \text{ are arbitrary} \right\}$$

(13.6) Meta Example II.

Assume we are given $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ in $\mathbb{R}^n$.

$V =$ all possible linear combinations of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$

$= \{ a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + a_m\vec{v}_m$ where $a_1, a_2, \ldots, a_m$ are real numbers $\}$

$\subset \mathbb{R}^n$ is a subspace.

- $\underline{\vec{0} \in V}$: take all $a_1 = a_2 = \ldots = a_m = 0$.

- If $\vec{u} = a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + a_m\vec{v}_m$
  $\vec{w} = b_1\vec{v}_1 + b_2\vec{v}_2 + \ldots + b_m\vec{v}_m$ are in V, then

$\vec{u} + \vec{w} = (a_1+b_1)\vec{v}_1 + (a_2+b_2)\vec{v}_2 + \ldots + (a_m+b_m)\vec{v}_m$ is in V. ✓

- Let $a \in \mathbb{R}$ and $\vec{u} = b_1\vec{v}_1 + b_2\vec{v}_2 + \ldots + b_m\vec{v}_m$ in V.

Then $a\vec{u} = (ab_1)\vec{v}_1 + (ab_2)\vec{v}_2 + \ldots + (ab_m)\vec{v}_m$ is still in V. ✓

e.g. $V =$ all possible linear combinations of $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$ $\subset \mathbb{R}^3$

This describes a plane in $\mathbb{R}^3$ containing $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$.

To get the equation, we find the normal vector by taking

$$\vec{n} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \times \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 1 & -1 & 2 \\ 2 & 0 & 3 \end{vmatrix}$$

$$= -3\vec{e}_1 + \vec{e}_2 + 2\vec{e}_3 = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$$

As $(0,0,0)$ lies on this plane, we get the equation as

$$-3(x-0) + 1(y-0) + 2(z-0) = 0$$

$$\boxed{-3x + y + 2z = 0}$$

So $V \subset \mathbb{R}^3$ can also be described as in Meta Example I:

$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ such that } \begin{bmatrix} -3 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \right\}$$