Lecture 14

(14.0) Recall - last time we introduced the notion of a <u>subspace</u> V of $\mathbb{R}^n$.

$V \subset \mathbb{R}^n$ (subset) is a <u>subspace</u> if
(C1) $\vec{0}$ is in V.
(C2) For any two $\vec{u}$ and $\vec{v}$ in V, the sum $\vec{u} + \vec{v}$ is in V.
(C3) For any vector $\vec{u}$ in V and a scalar $c \in \mathbb{R}$, $c\vec{u}$ is in V.

e.g. (i) $V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x - 2y + z = 4 \right\} \subset \mathbb{R}^3$ is <u>NOT</u> a subspace since $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is not in V. (C1) fails.

(ii) $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \geq y \right\} \subset \mathbb{R}^2$ is <u>NOT</u> a subspace.

$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is in V. Take scalar $c = -1 \in \mathbb{R}$. $\begin{bmatrix} -2 \\ -1 \end{bmatrix}$ is not in V
(C3) fails.

(14.1) We saw two type of examples of a subspace.

I. Pick A : $(m \times n)$-matrix. Consider $V =$ the set of <u>all solutions</u> to the homogeneous system $\boxed{A\vec{x} = \vec{0}}$.

Then $V \subset \mathbb{R}^n$ is a subspace.

II. Pick n vectors $\vec{v_1}, \vec{v_2}, \dots, \vec{v_n}$ in $\mathbb{R}^m$.

Let W = all linear combinations of $\vec{v_1}, \dots, \vec{v_n}$ $\subset \mathbb{R}^m$

Then W is a subspace.

Today - we are going to give names to these two important types of subspace.

<u>Note</u>: The input for both these examples is the same.
- namely an $(m \times n)$ matrix.

$\boxed{A : (m \times n) \text{ matrix}}$

$\xrightarrow{\text{I}}$ Look for $\vec{x}$ in $\mathbb{R}^n$ so that
$$A\vec{x} = \vec{0}$$

$\xrightarrow{\text{II}}$ Let $\vec{v_1}, \vec{v_2}, \dots, \vec{v_n}$ be the columns of A.

A more systematic way of putting these examples on equal footing is to view A as a "map" from $\mathbb{R}^n$ to $\mathbb{R}^m$

$$\vec{x} \in \mathbb{R}^n \rightsquigarrow \boxed{\begin{array}{c} A \\ m \times n \\ \text{matrix} \end{array}} \rightsquigarrow A \cdot \vec{x} \in \mathbb{R}^m$$

I. All $\vec{x}$ that go to $\vec{0} \in \mathbb{R}^m$

II. All $\vec{y}$ that come from $\boxed{A}$ applied to vectors from $\mathbb{R}^n$.

(14.2) Null space of a matrix (Type I examples).

Let A be an $(m \times n)$ matrix. The null space of A (sometimes called kernel of A), denoted by $\mathcal{N}(A)$, is defined as:

$$\mathcal{N}(A) = \{ \vec{x} \text{ in } \mathbb{R}^n \text{ such that } A\vec{x} = \underbrace{\vec{0}}_{\text{zero vector in } \mathbb{R}^m} \}$$

As we proved last time: $\mathcal{N}(A) \subset \mathbb{R}^n$ is a subspace.

e.g. (i) $A = O_{m \times n}$. (zero matrix)

$\mathcal{N}(O_{m \times n}) = \mathbb{R}^n$ since $O_{m \times n} \cdot \vec{x} = \vec{0}$ for every $\vec{x}$ in $\mathbb{R}^n$.

(ii) $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \end{bmatrix}_{2 \times 3}$

$A\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is same as

$$\begin{aligned} x_1 + 2x_3 &= 0 \\ x_2 - 2x_3 &= 0 \end{aligned} \quad \rightsquigarrow \quad \begin{aligned} x_1 &= -2x_3 \\ x_2 &= 2x_3 \end{aligned}$$

So $\mathcal{N}(A) = \left\{ x_3 \cdot \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \text{ where } x_3 \in \mathbb{R} \text{ is arbitrary} \right\} \subset \mathbb{R}^3$

(iii) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$ (identity matrix)

$\mathcal{N}(A) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} \subset \mathbb{R}^3$.

(14.3) Range of A (also called column span of A)

Again let A be an $m \times n$ matrix. The range of A, or column span of A, denoted by $R(A)$, is defined as:

$$R(A) = \left\{ \vec{y} \in \mathbb{R}^m \text{ such that } \vec{y} = A\vec{x} \text{ for some } \vec{x} \text{ in } \mathbb{R}^n \right\}$$

$$= \left\{ A\vec{x} \text{ where } \vec{x} \text{ in } \mathbb{R}^n \text{ is arbitrary} \right\} \subset \mathbb{R}^m$$

Recall- if $\vec{V_1}, \vec{V_2}, \ldots, \vec{V_n}$ are columns of A, that is:

$$A = \begin{bmatrix} a_{11} & a_{12} & & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} \leadsto \vec{V_1} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \vec{V_2} = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ldots,$$

$$\vec{V_n} = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

then
$$A\vec{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \end{bmatrix} = x_1\vec{v_1} + x_2\vec{v_2} + \ldots + x_n\vec{v_n}$$

So, $R(A) =$ all possible linear combinations of $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}$ $\subset \mathbb{R}^m$

(Type II example).

In order to describe range of a matrix, we are going to have to apply Gauss-Jordan method again.

That is, if we want to translate between type I and II examples, we will have to put A in a reduced echelon form.

For example, let us take a matrix in its reduced echelon form

$$A = \begin{bmatrix} 1 & 3 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{4 \times 5}.$$

$$\mathcal{N}(A) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \text{ such that } \begin{array}{l} x_1 = -3x_2 + 4x_5 \\ x_3 = 2x_5 \\ x_4 = x_5 \end{array} \right\} \quad \text{(type I - null space)}$$

$$= \left\{ x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 4 \\ 0 \\ 2 \\ 1 \\ 1 \end{bmatrix} \text{ where } x_2, x_5 \in \mathbb{R} \text{ are arbitrary} \right\}$$

$$= \text{all possible linear combinations of } \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 2 \\ 1 \\ 1 \end{bmatrix}.$$

$R(A) =$ all $\vec{b}$ in $\mathbb{R}^4$ such that $A\vec{x} = \vec{b}$ has a solution.

That is, $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ such that $A\vec{x} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ is consistent:

Augmented matrix $\left[\begin{array}{ccccc|c} 1 & 3 & 0 & 0 & -4 & b_1 \\ 0 & 0 & 1 & 0 & -2 & b_2 \\ 0 & 0 & 0 & 1 & -1 & b_3 \\ 0 & 0 & 0 & 0 & 0 & b_4 \end{array}\right]$ corresponds

to a consistent system if and only if $\boxed{b_4 = 0}$.

So $R(A) = \left\{ \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ 0 \end{bmatrix} \text{ where } b_1, b_2, b_3 \text{ are arbitrary} \right\} \subset \mathbb{R}^4.$

(14.4) Example. -(i) Let $A = \begin{bmatrix} 1 & -4 & 2 \\ -1 & 5 & 2 \\ 2 & -8 & 5 \end{bmatrix}_{3\times 3}$.

Describe the range of A.

Again $R(A)$ consists of all column vectors $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ so that $[A \mid \vec{b}]$ is consistent.

$$\left[\begin{array}{ccc|c} 1 & -4 & 2 & b_1 \\ -1 & 5 & 2 & b_2 \\ 2 & -8 & 5 & b_3 \end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 + R_1} \left[\begin{array}{ccc|c} 1 & -4 & 2 & b_1 \\ 0 & 1 & 4 & b_2 + b_1 \\ 0 & 0 & 1 & b_3 - 2b_1 \end{array}\right]$$

This system is always consistent.

So $\boxed{R(A) = \mathbb{R}^3}$.

(ii) Let $A = \begin{bmatrix} 1 & 0 & 4 \\ -1 & 1 & 7 \\ 1 & 1 & 15 \\ -1 & 2 & 18 \end{bmatrix}_{4\times 3}$. Describe the range $\mathcal{R}(A)$.

Again we are looking for b_1, b_2, b_3, b_4 so that $A\vec{x} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ is consistent.

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & b_1 \\ -1 & 1 & 7 & b_2 \\ 1 & 1 & 15 & b_3 \\ -1 & 2 & 18 & b_4 \end{array}\right] \xrightarrow[\substack{R_2 \to R_2 + R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 + R_1}]{} \left[\begin{array}{ccc|c} 1 & 0 & 4 & b_1 \\ 0 & 1 & 11 & b_2 + b_1 \\ 0 & 1 & 11 & b_3 - b_1 \\ 0 & 2 & 22 & b_4 + b_1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 4 & b_1 \\ 0 & 1 & 11 & b_2 + b_1 \\ 0 & 0 & 0 & (b_3 - b_1) - (b_2 + b_1) \\ 0 & 0 & 0 & (b_4 + b_1) - 2(b_2 + b_1) \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 4 & b_1 \\ 0 & 1 & 11 & b_2 + b_1 \\ \hline 0 & 0 & 0 & b_3 - b_2 - 2b_1 \\ 0 & 0 & 0 & b_4 - 2b_2 - b_1 \end{array}\right]$$

these two entries must be 0.

So the range of A consists of $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ such that $\boxed{\begin{array}{c} b_3 - b_2 - 2b_1 = 0 \\ \text{and} \\ b_4 - 2b_2 - b_1 = 0 \end{array}}$

So $\begin{array}{l} b_3 = 2b_1 + b_2 \\ b_4 = b_1 + 2b_2 \end{array}$. That is

$$\mathcal{R}(A) = \left\{ b_1 \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix} \right\}$$

$$= \left\{ \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \text{ such that } \begin{bmatrix} -2 & -1 & 1 & 0 \\ -1 & -2 & 0 & 1 \end{bmatrix} \vec{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$