Lecture 16

(16.0) Recall - Last time we defined what a basis of a subspace is.

- Subspace. – $V \subset \mathbb{R}^N$ is a <u>subspace</u> if

  (C1) $\vec{0} \in V$. (C2) for any two $\vec{u}, \vec{v}$ in V; $\vec{u}+\vec{v}$ is in V.

  (C3) for $\vec{u}$ in V and $c \in \mathbb{R}$ (scalar); $c\vec{u}$ is in V.

- Let $B = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$ be a <u>set of vectors from V</u>

  We say B is <u>a basis of V</u> if

  $\boxed{\text{Span}(B) = V}$ Meaning: every $\vec{w}$ in V, can be written as

  $$\vec{w} = a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_p\vec{v}_p.$$

  (read: B <u>spans</u> V.)

  $\boxed{B \text{ is linearly independent}}$ Meaning: if $\vec{0} = x_1\vec{v}_1 + x_2\vec{v}_2 + \dots + x_p\vec{v}_p$

  then $x_1 = x_2 = \dots = x_p = 0$.

We saw last time that the two conditions written above, for $B \subset V$ to be a basis can be put together into one:

$$\boxed{\text{Every } \vec{w} \text{ in } V \text{ can be written } \underline{\text{uniquely}} \text{ as } \vec{w} = a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_p\vec{v}_p}$$

(16.1) We proved that if $B \subset V$ is <u>a basis</u> consisting of p vectors $(B = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\})$, then

- any set $\{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_q\}$ containing $q > p$ vectors from V has to be linearly ~~in~~dependent.
- any other basis $B' \subset V$ will also have exactly p elements.

We defined $\boxed{\dim(V) = p}$.
(dimension of V)

Summary. — Basis is NOT unique (a subspace $V \subset \mathbb{R}^N$ can have many bases) (does have infinitely many) ; but number of vectors in a basis is the same for all of them $= \dim(V)$.

(16.2) Example. Consider the following 4 vectors in $\mathbb{R}^3$:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}.$$

Let $V = \mathrm{Span}(\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}) \subset \mathbb{R}^3$.

Find a basis of V.

$\rightarrow$ I am going to solve this problem using 3 methods.

We saw these methods earlier – (I) Example on page 7 of Lecture 14.
(II) Example (iv) - page 3 of Lecture 15.
(III) Example on page 8 of Lecture 15.

Method I :

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \text{ is in } V \overset{\text{(same as)}}{\longleftrightarrow} x_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

can be solved

i.e. $\left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & b_1 \\ 0 & 1 & 1 & 1 & b_2 \\ -1 & 2 & 4 & 3 & b_3 \end{array}\right]$ is the augmented matrix of a consistent system.

Put it in reduced echelon form:

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & b_1 \\ 0 & 1 & 1 & 1 & b_2 \\ -1 & 2 & 4 & 3 & b_3 \end{array}\right] \xrightarrow{R_3 \to R_3 + R_1} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & b_1 \\ 0 & 1 & 1 & 1 & b_2 \\ 0 & 4 & 4 & 4 & b_3 + b_1 \end{array}\right] \xrightarrow{R_3 \to R_3 - 4R_2} \underbrace{\left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & b_1 \\ 0 & 1 & 1 & 1 & b_2 \\ 0 & 0 & 0 & 0 & b_3 + b_1 - 4b_2 \end{array}\right]}_{\text{echelon form}}$$

$$\xrightarrow{R_1 \to R_1 - 2R_2} \left[\begin{array}{cccc|c} 1 & 0 & -2 & -1 & b_1 - 2b_2 \\ 0 & 1 & 1 & 1 & b_2 \\ 0 & 0 & 0 & 0 & b_3 + b_1 - 4b_2 \end{array}\right]$$

Consistent if and only if $\boxed{b_3 + b_1 - 4b_2 = 0}$ "algebraic description of V"

So $V = \left\{ \begin{bmatrix} b_1 \\ b_2 \\ -b_1 + 4b_2 \end{bmatrix} ; \text{ where } b_1, b_2 \in \mathbb{R} \text{ are arbitrary} \right\}$

$= \left\{ b_1 \cdot \underbrace{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}_{\vec{v}_1} + b_2 \underbrace{\begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}}_{\vec{v}_3} ; \text{ where } b_1, b_2 \in \mathbb{R} \text{ are arbitrary} \right\}$

Thus $\{\vec{v}_1, \vec{v}_3\}$ spans V. It is easily seen to be linearly independent, hence a basis of V.

$$\left(x_1\vec{v}_1 + x_3\vec{v}_3 = \begin{bmatrix} x_1 \\ x_2 \\ -x_1+4x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ implies } x_1=0 \;\&\; x_2=0 \right)$$

Method II. Find linear dependence relations among $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ and throw away those which can be expressed as a linear combination of others – one by one.

$$x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 + x_4\vec{v}_4 = \vec{0}$$

$$\longleftrightarrow \quad \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ -1 & 2 & 4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We already did the work to put the coefficient matrix of this homogeneous system into reduced echelon form.

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ -1 & 2 & 4 & 3 \end{bmatrix} \xrightarrow{\text{see last page}} \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So $x_1 = 2x_3 + x_4$ $\qquad$ x_1, x_2 – dependent variables

$x_2 = -x_3 - x_4$ $\qquad$ x_3, x_4 – free parameters.

→ Specialize free parameters "cleverly" -

- set all to 0, except for one.

$x_3 = 1,\ x_4 = 0 \rightsquigarrow \begin{matrix} x_1 = 2 \\ x_2 = -1 \end{matrix} \rightsquigarrow 2\vec{v}_1 - \vec{v}_2 + \vec{v}_3 = \vec{0}$

same

$\vec{v}_3 = -2\vec{v}_1 + \vec{v}_2$

(so $\vec{v}_3$ is redundant)

$x_3 = 0,\ x_4 = 1 \rightsquigarrow \begin{matrix} x_1 = 1 \\ x_2 = -1 \end{matrix} \rightsquigarrow \vec{v}_1 - \vec{v}_2 + \vec{v}_4 = 0$

$\vec{v}_4 = -\vec{v}_1 + \vec{v}_2$

(so $\vec{v}_4$ is redundant).

- from $\vec{v}_3 = -2\vec{v}_1 + \vec{v}_2$ & $\vec{v}_4 = -\vec{v}_1 + \vec{v}_2$ we see that $\text{Span}(\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}) = \text{Span}(\{\vec{v}_1, \vec{v}_2\})$

- Why are $\{\vec{v}_1, \vec{v}_2\}$ linearly independent?

Notice we already did the work - focus on Cols 1 & 2 of the matrix on pages 3 & 4.
(3×4)

$x\vec{v}_1 + y\vec{x}_2 = \vec{0} \longleftrightarrow \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$; $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -1 & 2 \end{bmatrix} \xrightarrow[\text{form}]{\text{reduced echelon}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

i.e. $x = 0$ & $y = 0$ ✓

Hence, $\{\vec{v}_1, \vec{v}_2\}$ is a basis of V.

Method III. • Write $\vec{V_1}, \vec{V_2}, \vec{V_3}, \vec{V_4}$ as <u>ROWS</u> of a matrix

$$B = \begin{array}{c} \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 2 \\ 0 & 1 & 4 \\ 1 & 1 & 3 \end{bmatrix} \end{array} \begin{array}{l} \leftarrow V_1^T \\ \leftarrow V_2^T \\ \leftarrow V_3^T \\ \leftarrow V_4^T \end{array}$$

(Recall: A^T (transpose of A) is defined by $\underbrace{(A^T)}_{n\times m}{}_{ij} = \underbrace{(A)}_{m \times n}{}_{ji}$)

• Put B in reduced echelon form

$$B = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 2 \\ 0 & 1 & 4 \\ 1 & 1 & 3 \end{bmatrix} \xrightarrow[R_4 \to R_4 - R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 4 \\ 0 & 1 & 4 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow[R_4 \to R_4 - R_1]{R_3 \to R_3 - R_1} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = RE(B)$$

→ Non-zero rows of $\underbrace{\cancel{B}}_{RE(B)}$ give a basis of the row span of B.

So, V has a basis $\vec{u_1} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\vec{u_2} = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$.

(16.3) Why does method III work?*

We need to verify that row operations do not change the span of row vectors.

Let $A = \begin{bmatrix} — \vec{V_1}^T — \\ — \vec{V_2}^T — \\ \vdots \\ — \vec{V_m}^T — \end{bmatrix}$; Row span of $A = \mathrm{Span}(\{\vec{V_1}, \dots, \vec{V_m}\})$

and let $B = \begin{bmatrix} — \vec{W_1}^T — \\ — \vec{W_2}^T — \\ \vdots \\ — \vec{W_m}^T — \end{bmatrix}$ obtained from A after performing one elementary row operation.

Row span of $B = \mathrm{Span}\{\vec{W_1}, \vec{W_2}, \dots, \vec{W_m}\}$

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*Optional

$$\text{Span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\} \overset{?}{=} \text{Span}\{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_m\}$$

Row Operations. - (1) <u>Swap.</u> If $A \xrightarrow[R_i \leftrightarrow R_j]{} B$, then

$\vec{w}_1 = \vec{v}_1, \dots, \boxed{\begin{matrix}\vec{w}_i = \vec{v}_j \\ \vec{w}_j = \vec{v}_i\end{matrix}}, \dots$ so clearly they span the same subspace.

(2) <u>Scale</u> If $A \xrightarrow[R_i \to \alpha R_i]{} B$ where $\alpha \neq 0$. So $\vec{w}_i = \alpha \vec{v}_i$
$\vec{w}_j = \vec{v}_j \ (j \neq i)$

Again it is clear that span of $\vec{v}_1, \dots, \vec{v}_m$ = Span of $\vec{w}_1, \dots, \vec{w}_m$.

(3) <u>Combine</u> If $i \neq j$; $A \xrightarrow[R_i \to R_i - \lambda R_j]{} B$,
$\lambda \in \mathbb{R}$

$\vec{w}_i = \vec{v}_i - \lambda \vec{v}_j$ all other $\vec{w}_k = \vec{v}_k \ (k \neq i)$.

Linear Combination of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$		Linear combination of $\vec{w}_1, \vec{w}_2, \dots, \vec{w}_m$
$a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_m\vec{v}_m$	$\longleftrightarrow$ equal to	$a_1\vec{w}_1 + \dots + a_i\vec{w}_i + \dots + (a_j + \lambda a_i)\vec{w}_j + \dots + a_m\vec{w}_m$
$b_1\vec{v}_1 + \dots + b_i\vec{v}_i + \dots + (b_j - \lambda b_i)\vec{v}_j + \dots + b_m\vec{v}_m$	$\longleftrightarrow$ equal to	$b_1\vec{w}_1 + b_2\vec{w}_2 + \dots + b_m\vec{w}_m$

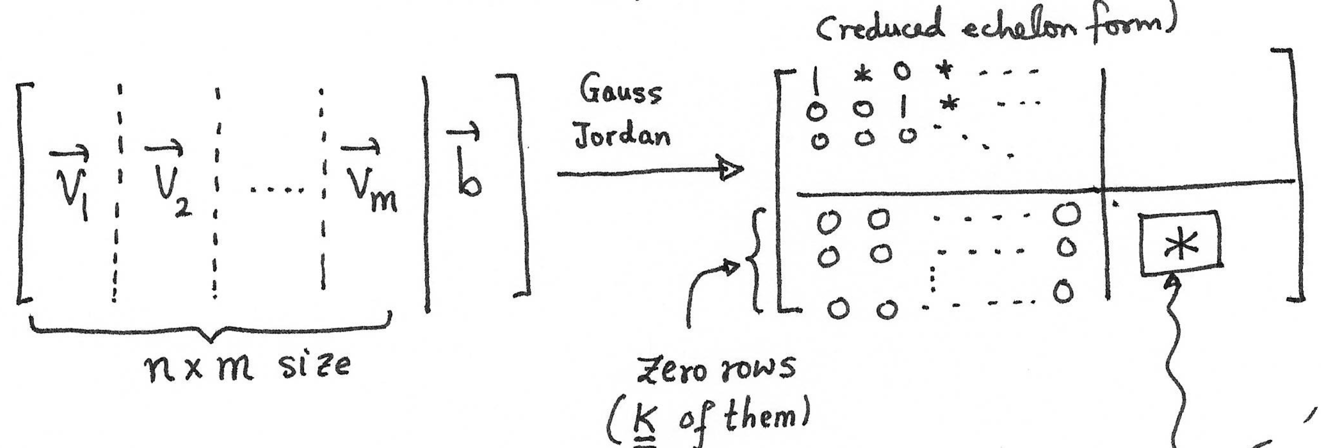
So, they are equal.

(16.4) Summary of these methods.

Let $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\}$ consist of m vectors in $\mathbb{R}^n$.

Let $V = \text{Span}(S) \subset \mathbb{R}^n$ subspace.

Method I is used if we want to find an <u>algebraic description</u> of V. Meaning, we want to find a matrix C ($k \times n$) so that <u>$V =$ Null Space of C</u> $= \{\vec{x} \in \mathbb{R}^n \mid C\vec{x} = \vec{0}\}$.



$$\begin{array}{c}\text{Algebraic}\\\text{Description}\\\text{of } V\end{array} = \left[\begin{array}{l}k \text{ linear expressions}\\\text{involving } b_1, b_2, \dots, b_n\\= 0\end{array}\right]$$

Method II is used if we want to find <u>a subset of</u> $S = \{\vec{v}_1, \dots, \vec{v}_m\}$ which is a <u>basis of V</u>.

— next page →

Method II continued – We find linear dependence among $\vec{V}_1, \dots, \vec{V}_m$ and throw away redundant vectors "cleverly".

$$\underbrace{\left[\begin{array}{c:c:c:c} \vec{V}_1 & \vec{V}_2 & \cdots & \vec{V}_m \end{array}\right]}_{n \times m} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \longleftrightarrow$$

a typical dependence relation $x_1\vec{V}_1 + x_2\vec{V}_2 + \dots + x_m\vec{V}_m = \vec{0}$

$\Big\downarrow$ Gauss-Jordan

$$\begin{bmatrix} 1 & * & 0 & 0 & * & \cdots \\ 0 & 0 & 1 & 0 & * & \cdots \\ 0 & 0 & 0 & 1 & \ddots & \\ \hline & & \text{Zero Rows} & & & \end{bmatrix}$$

(reduced echelon form)

- free parameters $\leftrightarrow$ Columns <u>without</u> pivotal 1's.
- Throw away vectors (from S) whose <u>subscript</u> <u>matches</u> those of <u>free parameters</u>

(e.g. if x_2, x_5, x_6 are free parameters & $S = \{\vec{V}_1, \vec{V}_2, \vec{V}_3, \dots, \vec{V}_8\}$ $\rightsquigarrow$ $\{\vec{V}_1, \vec{V}_3, \vec{V}_4, \vec{V}_7, \vec{V}_8\}$ BASIS)

$\boxed{\text{Method III}}$ is used if we want to <u>just find a basis</u> of V.

Put $\vec{V}_1, \dots, \vec{V}_m$ as <u>ROWS</u> of a matrix

$$\begin{bmatrix} \text{——} & \vec{V}_1^T & \text{——} \\ \text{——} & \vec{V}_2^T & \text{——} \\ & \vdots & \\ \text{——} & \vec{V}_m & \text{——} \end{bmatrix} \xrightarrow{\text{Gauss Jordan}} \begin{array}{r} \\ \\ \text{zero rows} \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. \end{array} \begin{bmatrix} \text{——} & u_1^T & & \text{——} \\ & \vdots & & \\ \text{——} & u_\ell^T & & \text{——} \\ \hline 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{array}{l} \left. \begin{array}{c} \\ \\ \end{array} \right\} \text{non-zero rows} \\ \\ \\ \\ \end{array}$$

(Reduced echelon form)

<u>Answer</u> – $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_\ell\}$ is a basis of V.

(16.5) Rank and nullity of an $(m \times n)$ matrix A.

Rank(A) is defined as the dimension of $\mathcal{R}(A)$ = range of A
$= \text{Span}(\text{Columns of } A)$.

Nullity(A) is defined as the dimension of null space of A

$$\mathbb{R}^n \dashrightarrow \boxed{\begin{matrix} A : (m \times n) \\ \text{matrix} \end{matrix}} \dashrightarrow \mathbb{R}^m$$

$$\vec{x} \dashrightarrow \qquad\qquad \dashrightarrow \vec{y} = A\vec{x}$$

$\mathcal{N}(A) \subset \mathbb{R}^n$ $\qquad\qquad$ $\mathcal{R}(A) \subset \mathbb{R}^m$

$\boxed{\dim(\mathcal{N}(A)) = \text{nullity of } A}$ $\qquad\qquad$ $\boxed{\dim(\mathcal{R}(A)) = \text{rank of } A}$

$\boxed{\text{Method II}}$ for finding a basis of Span(Columns of A):

$$\underset{(m \times n)}{A} \dashrightarrow \mathcal{RE}(A) = \left[\begin{matrix} 1 & * & 0 & 0 & * & \cdots \\ 0 & 0 & 1 & 0 & * & \cdots \\ 0 & 0 & 0 & 1 & & \ddots \\ \hline & & \text{zero rows} & & & \end{matrix}\right] \rightsquigarrow$$

Columns of A corresponding to $\boxed{\text{Columns of } \mathcal{RE}(A) \text{ containing pivotal 1's}}$ is a basis of $\mathcal{R}(A)$

So $\dim(\mathcal{R}(A)) = \#$ of pivotal 1's
$= n - (\#\text{ free parameters})$

But $\#$ Free parameters $= \dim(\mathcal{N}(A))$.

$$\boxed{n = \dim(\mathcal{R}(A)) + \dim(\mathcal{N}(A))} \quad \begin{pmatrix} \text{RANK-NULLITY} \\ \text{Theorem} \end{pmatrix}$$

$\boxed{\text{Method III}}$ implies $\boxed{\text{rank}(A) = \text{rank}(A^T)}$

(proof is optional – see textbook's section 3.5 THEOREM 10 (pages 209, 211) if you are interested.)