Lecture 19

(19.0) Recall: last time we defined <u>linear transformations</u>.

<u>Definition</u> Let $V \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^m$ be two subspaces.

A function $T : V \to W$ is called <u>a linear transformation</u>

if (1) for every $\vec{v}_1, \vec{v}_2$ in V, we have

$$T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2)$$

(2) for every $\vec{v}$ in V and a scalar $c \in \mathbb{R}$

$$T(c\vec{v}) = cT(\vec{v})$$

In words: T must respect <u>vector addition</u> and <u>scalar multiplication</u>.

e.g. $V = \mathbb{R}^2$; $W = \mathbb{R}^2$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by the following formula

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x - y \\ x + y \end{bmatrix}.$$

Then T is a linear transformation.

<u>Check</u>: (1) $T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right)$

Left-hand side : $T\left(\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}\right) = \begin{bmatrix} 2(x_1 + x_2) - (y_1 + y_2) \\ (x_1 + x_2) + (y_1 + y_2) \end{bmatrix}$

$= \begin{bmatrix} 2x_1 - y_1 \\ x_1 + y_1 \end{bmatrix} + \begin{bmatrix} 2x_2 - y_2 \\ x_2 + y_2 \end{bmatrix}$ = Right-hand side ✓

(2) $T\left(\begin{bmatrix} cx \\ cy \end{bmatrix}\right) = c\,T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$

Left-hand side: $T\left(\begin{bmatrix} cx \\ cy \end{bmatrix}\right) = \begin{bmatrix} 2(cx)-(cy) \\ (cx)+(cy) \end{bmatrix} = c\begin{bmatrix} 2x-y \\ x+y \end{bmatrix}$

$= c\,T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ = Right-hand side ✓

- Let us compute T on some vector to get better acquinted:

$$T\left(\begin{bmatrix} 1 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 2(1)-4 \\ 1+4 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}.$$

- find a vector $\begin{bmatrix} x \\ y \end{bmatrix}$ so that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x-y \\ x+y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ translates to the following

linear system $\begin{aligned} 2x-y &= 1 \\ x+y &= 1 \end{aligned}$. Solution: $\begin{aligned} x &= \frac{2}{3} \\ y &= \frac{1}{3} \end{aligned}$.

(19.1) Last time I mentioned that linear transformations

$$\mathbb{R}^n \longrightarrow \mathbb{R}^m$$

are given by m functions of n variables each:

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \rightsquigarrow \begin{bmatrix} f_1(x_1, x_2, \dots, x_n) \\ f_2(x_1, x_2, \dots, x_n) \\ \vdots \\ f_m(x_1, x_2, \dots, x_n) \end{bmatrix}.$$

And every f_j must be a <u>linear expression</u> without <u>constant term</u>.

e.g. $\begin{bmatrix} x \\ y \end{bmatrix} \rightsquigarrow \begin{bmatrix} x+y+1 \\ 2y \end{bmatrix}$ is <u>NOT</u> a linear transformation

$(\mathbb{R}^2 \longrightarrow \mathbb{R}^2)$

A quick way to see this is to use the following property of linear transformations:

$T: V \to W$ linear transformation. Then $\boxed{T(\vec{0}) = \vec{0}}$.

(Proof. Since $T(\vec{v_1} + \vec{v_2}) = T(\vec{v_1}) + T(\vec{v_2})$, taking $\vec{v_1} = \vec{v_2} = \vec{0}$ gives

$$T(\vec{0} + \vec{0}) = T(\vec{0}) + T(\vec{0})$$

$\left(= T(\vec{0}) \right.$ because $\left. \vec{0} + \vec{0} = \vec{0} \right)$

So, $T(\vec{0}) = T(\vec{0}) + T(\vec{0})$.

Subtracting $T(\vec{0})$ from both sides gives $T(\vec{0}) = \vec{0}$.)

Now in the function given above $F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y+1 \\ 2y \end{bmatrix}$,

$x = y = 0$ gives $F\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

So, F is not a linear transformation.

(19.2) Warm up: every linear transformation $\mathbb{R} \xrightarrow{T} \mathbb{R}$ is given by fixing a number $\alpha \in \mathbb{R}$ and

$$T(x) = \alpha x \quad \text{for every } x \in \mathbb{R}$$

Let us first check that if $\alpha \in \mathbb{R}$ is fixed and we have

$$T : \mathbb{R} \to \mathbb{R} \text{ given by } T(x) = \alpha x$$

then T is a linear transformation.

(1) $T(x_1 + x_2) = \alpha(x_1 + x_2) = \alpha x_1 + \alpha x_2 = T(x_1) + T(x_2)$ ✓

(2) $T(cx) = \alpha(cx) = c(\alpha x) = cT(x)$ ✓

Now if $T : \mathbb{R} \to \mathbb{R}$ is <u>any</u> <u>linear transformation</u>, take $\alpha = T(1)$.

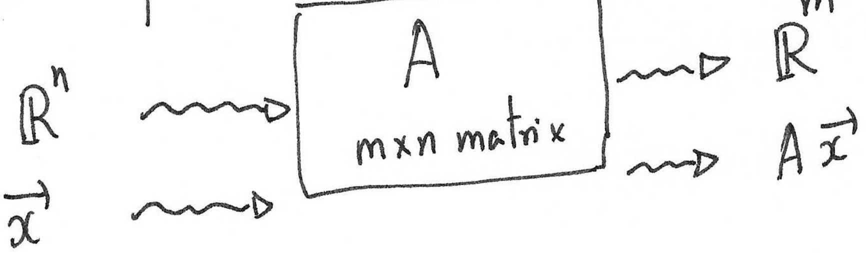
by $T(cx) = cT(x)$ property, we get - by taking $x = 1$ - that $T(c) = cT(1) = c\alpha$ for every $c \in \mathbb{R}$.

So, T is of the form $c \in \mathbb{R} \longmapsto \alpha c \in \mathbb{R}$.

(19.3) Main example: let A be an $m \times n$ matrix.

Define $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ by $T(\vec{x}) = A\vec{x}$

(remember the picture



)

Then T is a linear transformation.

Check (1) $T(\vec{x}_1+\vec{x}_2) = T(\vec{x}_1) + T(\vec{x}_2)$

Left-hand side: $\underset{m\times n}{A}\,(\underset{n\times 1}{\vec{x}_1+\vec{x}_2}) = A\vec{x}_1 + A\vec{x}_2$ (matrix multiplication distributes over addition)

$= T(\vec{x}_1) + T(\vec{x}_2)$ = Right-hand side. ✓

(2) $T(c\vec{x}) = cT(\vec{x})$

Left-hand side $= A(c\vec{x}) = c(A\vec{x})$ (because c is a scalar)

$= cT(\vec{x})$ = Right-hand side ✓

e.g. (i) $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \end{bmatrix}_{2\times 3}$ and $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_3 \\ -x_2 + 3x_3 \end{bmatrix}.$$

(ii) Let $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 5x - 3y \\ 2x \\ 7y \end{bmatrix}.$$ Then

$T(\vec{x}) = A\vec{x}$ for the following 3×2 matrix $\begin{bmatrix} 5 & -3 \\ 2 & 0 \\ 0 & 7 \end{bmatrix}$.

(19.4) Linear Transformations $\leadsto$ Matrices.

Let $V \subset \mathbb{R}^n$ be a subspace of dimension p.

Let us choose a basis $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$.

Then any linear transformation $T : V \to W$ $\left(\begin{array}{c} W \subset \mathbb{R}^m \\ \text{another subspace} \end{array}\right)$

is completely determined by where it sends $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$.

Let us write
$$\left.\begin{array}{l} \vec{b}_1 = T(\vec{v}_1) \\ \vec{b}_2 = T(\vec{v}_2) \\ \quad \vdots \\ \vec{b}_p = T(\vec{v}_p) \end{array}\right\} \text{ p vectors in } W.$$

Given any vector $\vec{v}$ in V, there is a unique way of expressing $\vec{v}$ as a linear combination of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$

(since $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$ is a basis)

$$\vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_p\vec{v}_p. \quad (a_1, \dots, a_p \text{ are numbers})$$

Then by properties (1) and (2) of linear transformations

$$T(\vec{v}) = T\left(a_1\vec{v}_1 + (a_2\vec{v}_2 + \dots + a_p\vec{v}_p)\right)$$
$$= a_1 T(\vec{v}_1) + T(a_2\vec{v}_2 + \dots + a_p\vec{v}_p) = \dots$$
$$\dots = a_1 T(\vec{v}_1) + a_2 T(\vec{v}_2) + \dots + a_p T(\vec{v}_p)$$

$$= a_1 \vec{b}_1 + a_2 \vec{b}_2 + \dots + a_p \vec{b}_p .$$

e.g. Let $V = \text{Span}\left\{ \vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\} \subset \mathbb{R}^3$

(Note $\{\vec{v}_1, \vec{v}_2\}$ are linearly independent, hence a basis of V)

A linear transformation $T : V \longrightarrow \mathbb{R}^2$ can be defined by:

$$T(\vec{v}_1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad T(\vec{v}_2) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

To compute $T(\vec{v})$ for some $\vec{v} \in V$, we will first have to express it as a linear combination of $\vec{v}_1$ and $\vec{v}_2$.

For instance: $\vec{v} = \begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix} = 5\vec{v}_1 + 4\vec{v}_2$.

$$\text{So } T(\vec{v}) = 5\,T(\vec{v}_1) + 4\,T(\vec{v}_2)$$

$$= 5 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 4 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 5+8 \\ 10+16 \end{bmatrix} = \begin{bmatrix} 13 \\ 26 \end{bmatrix}.$$

(19.5) $\boxed{\text{Linear Transformations } \mathbb{R}^n \longrightarrow \mathbb{R}^m} \longleftrightarrow \boxed{m \times n \text{ matrices}}$

If $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is any linear transformation, then by the result of previous section, we know T is completely determined by its value on $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$.

$$\left(\text{Recall: } \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ \ldots, \ \vec{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \ \text{coordinate basis of } \mathbb{R}^n. \right)$$

$$\underbrace{\vec{b}_1 = T(\vec{e}_1), \ \vec{b}_2 = T(\vec{e}_2), \ \ldots, \ \vec{b}_n = T(\vec{e}_n)}_{n \text{ vectors in } \mathbb{R}^m \text{ - so } m \times 1 \text{ - columns}}$$

$\leadsto$ we get an $m \times n$ matrix $A = \begin{bmatrix} \vec{b}_1 & \vdots & \vec{b}_2 & \vdots & \cdots & \vdots & \vec{b}_n \end{bmatrix}_{m \times n}$.

Check: $T(\vec{x}) = A\vec{x}$ for every $\vec{x}$ in $\mathbb{R}^n$.

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \vec{e}_1 + x_2 \vec{e}_2 + \ldots + x_n \vec{e}_n .$$

We get : $T(\vec{x}) = x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2) + \ldots + x_n T(\vec{e}_n)$

from the two properties of linear transformations

$$= x_1 \vec{b}_1 + x_2 \vec{b}_2 + \ldots + x_n \vec{b}_n .$$

Similarly - as we have seen many times by now

$$A\vec{x} = x_1 \vec{b}_1 + x_2 \vec{b}_2 + \ldots + x_n \vec{b}_n \qquad \square$$