Lecture 20

(20.0) Recall: given $V \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^m$ two subspaces, a linear transformation from V to W is a function

$$T : V \longrightarrow W$$

such that (1) $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2)$ for every $\vec{v}_1, \vec{v}_2 \in V$.

(2) $T(c\vec{v}) = c\,T(\vec{v})$ for every $\vec{v} \in V$ and $c \in \mathbb{R}$.

We proved that a linear transformation always sends $\vec{0} \in V$ to $\vec{0} \in W$.

$$\boxed{T(\vec{0}) = \vec{0}}$$

We also showed that linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$ are nothing but $m\times n$ matrices

$$\boxed{\begin{matrix}\text{Linear Transformations}\\ \mathbb{R}^n \to \mathbb{R}^m\end{matrix}} \longleftrightarrow \boxed{m\times n \text{ matrices}}$$

- Given A : an $m\times n$ matrix, define the linear transformation it corresponds to $T : \mathbb{R}^n \to \mathbb{R}^m$ by:

$$T(\vec{x}) = A\vec{x}$$

- Given a linear transformation $F : \mathbb{R}^n \to \mathbb{R}^m$, build the matrix of F by

$$\left[\, F(\vec{e}_1) \;\vdots\; F(\vec{e}_2) \;\vdots\; \cdots \;\vdots\; F(\vec{e}_n) \,\right]$$

here $\{\vec{e}_1, \dots, \vec{e}_n\}$ is the coordinate basis of $\mathbb{R}^n$.

(20.1) Null space and range of a linear transformation are defined analogous to their definition for matrices:

Let $T : V \to W$ be a linear transformation.

Null space of $T = \mathcal{N}(T) = \{\vec{v} \text{ in } V \text{ such that } T(\vec{v}) = \vec{0}\}$.

Range of $T = \mathcal{R}(T) = \{\vec{w} \text{ in } W \text{ such that } \vec{w} = T(\vec{v}) \text{ for some } \vec{v} \text{ in } V\}$.

Nullity (T) = dimension of $\mathcal{N}(T) \subset V$.

Rank (T) = dimension of $\mathcal{R}(T) \subset W$.

Later in this course we will reprove the rank-nullity theorem:

$$\boxed{\text{Nullity}(T) + \text{Rank}(T) = \dim(V)}$$

(20.2) Just like for $\mathbb{R}^n$ and $\mathbb{R}^m$, a linear transformation $T : V \to W$ (here $V \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^m$ are subspaces) can always be represented as a matrix.

Very Important - The representation of a linear transformation via a matrix depends on a choice of a basis of V and a basis of W.

For $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$ - we had a "natural choice" - $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ for $\mathbb{R}^n$ and $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_m\}$ for $\mathbb{R}^m$.

How does it work?

Let $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\} \subset V$ be a basis of V (so, $\dim(V) = p$).

Let $\{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_q\} \subset W$ be a basis of W (so $\dim(W) = q$).

Given a linear transformation $T : V \to W$, we can build a $q \times p$ size matrix as follows:

$T(\vec{v}_1) \in W$. So we can express it as a linear combination

$$T(\vec{v}_1) = a_{11}\vec{w}_1 + a_{21}\vec{w}_2 + \dots + a_{q1}\vec{w}_q$$

Similarly,

$$T(\vec{v}_2) = a_{12}\vec{w}_1 + a_{22}\vec{w}_2 + \dots + a_{q2}\vec{w}_q$$

$$\vdots$$

$$T(\vec{v}_p) = a_{1p}\vec{w}_1 + a_{2p}\vec{w}_2 + \dots + a_{qp}\vec{w}_q$$

$$\rightsquigarrow \quad A = \begin{bmatrix} a_{11} & a_{12} & & a_{1p} \\ a_{21} & a_{22} & \dots & a_{2p} \\ \vdots & \vdots & & \vdots \\ a_{q1} & a_{q2} & & a_{qp} \end{bmatrix}$$

$q \times p$ size matrix.

How does A encode the linear transformation we started from?

1. Given $\vec{v} \in V$; express $\vec{v}$ as a linear combination of $\{\vec{v}_1, \dots, \vec{v}_p\}$

$$\vec{v} = x_1\vec{v}_1 + x_2\vec{v}_2 + \dots + x_p\vec{v}_p.$$

2. Multiply A by $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}$ to get $\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_q \end{bmatrix} = A \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}$

3. $T(\vec{v}) = y_1 \vec{w}_1 + y_2 \vec{w}_2 + \dots + y_q \vec{w}_q$.

<u>Summary</u>:

$$\boxed{\begin{matrix} \text{Linear Transformations} \\ V \to W \end{matrix}} \longleftrightarrow \boxed{\begin{matrix} q \times p \text{ size} \\ \text{matrices} \end{matrix}}$$

$(p = \dim(V)\,;\ q = \dim(W))$

once a basis of V and a basis of W have been chosen

(20.3) Example. $V = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\} \subset \mathbb{R}^3$

$$W = \mathbb{R}^2.$$

$T : V \longrightarrow W$ given by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y \\ y+z \end{bmatrix}$.

- Basis of $V = \left\{ \underset{\vec{v}_1}{\underset{\shortparallel}{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}}, \underset{\vec{v}_2}{\underset{\shortparallel}{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}} \right\}$, Basis of $W = \left\{ \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

$T(\vec{v}_1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \vec{e}_1 + \vec{e}_2$

$T(\vec{v}_2) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \vec{e}_1 + 2\vec{e}_2$

$\rightsquigarrow$ Matrix representing T (with this choice of bases)

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

- Let us change basis of V to $\left\{ \vec{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}$

Keep the same basis for W : $\left\{ \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{e}_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

$T(\vec{u}_1) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ $T(\vec{u}_2) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ $\rightsquigarrow$ Matrix representing T (with this new choice of bases)

$= 0 \cdot \vec{e}_1 + (-1)\vec{e}_2$ $\quad = 2\vec{e}_1 + 3\vec{e}_2$

$$\begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix}.$$

Let $\vec{v} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} \in V$. So, from the formula of T

$$T(\vec{v}) = \begin{bmatrix} 2+2 \\ 2+4 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

$\vec{v} = 2\vec{v}_1 + 2\vec{v}_2$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix} \checkmark$$

matrix representing T when basis of V is $\{\vec{v}_1, \vec{v}_2\}$

$\vec{v} = 2\vec{u}_2 = 0 \cdot \vec{u}_1 + 2\vec{u}_2$

$$\begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix} \checkmark$$

matrix representing T when basis of V is chosen to be $\{\vec{u}_1, \vec{u}_2\}$