Lecture 21

(21.0) Recall - So far we considered $\mathbb{R}^N$ and its subspaces.

Two main operations on $\mathbb{R}^N$: <u>addition</u> and <u>scalar multiplication</u>.

A subspace $V \subset \mathbb{R}^N$ was any subset containing $\vec{0}$ and "closed under <u>addition</u> and <u>scalar multiplication</u>".

Also recall that <u>addition</u> and <u>scalar multiplication</u> satisfy a list of properties: (see Lecture 13, page 2).

There are many mathematical objects which admit there two operations and satisfy the properties listed in lecture 13. It seems reasonable to call those objects <u>vector spaces</u> as well.

For example. - Consider the set of all functions of one variable.

$$\mathbb{F} = \{ f : \mathbb{R} \to \mathbb{R} \}.$$

(e.g. $\sin(x)$, $\cos(x)$, e^x, $x^4 + x^2 + 1$ ... are all "elements" of $\mathbb{F}$).

- functions can be added together to get another function
- functions can be scaled by a real number

(e.g. $\sin(x) + e^x + \cos(4x)$ - another function of one variable

$\sqrt{2} \cdot \sin(x)$; $4 \cdot e^x$; $\frac{1}{9} \cdot \cos(2x)$ - scales of $\sin(x)$; e^x; $\cos(2x)$)

Example: Solutions of homogeneous differential equations

$$\{ f(x) \text{ such that } f''(x) = -f(x) \}.$$

- if $f'' = -f$ and $g'' = -g$ then $(f+g)'' = f'' + g'' = -(f+g)$.
- if $f'' = -f$ and $c \in \mathbb{R}$ then $(cf)'' = c \cdot f'' = -(cf)$.

e.g. $\frac{d}{dx} \sin(x) = \cos(x)$; $\frac{d^2}{dx^2} \sin(x) = \frac{d}{dx} \cos(x) = -\sin(x)$.

$\frac{d}{dx} \cos(x) = -\sin(x)$; $\frac{d^2}{dx^2} \cos(x) = -\frac{d}{dx} \sin(x) = -\cos(x)$.

any function of the form $a \sin(x) + b \cos(x) = f(x)$ also ($a, b \in \mathbb{R}$ fixed)

satisfies $f''(x) = -f(x)$.

In this part of the course we are going to define abstract vector spaces and all the previously studied concepts (subspace, basis, linear independence, Span, linear transformations ...) in this abstract setting.

(21.1) <u>Vector space</u>. A vector space (over $\mathbb{R}$) is a <u>set</u> V on which 2 operations are defined:

- addition. $$V \times V \longrightarrow V$$ $$(\vec{v}_1, \vec{v}_2) \longmapsto \vec{v}_1 + \vec{v}_2$$

(we can add two <u>vectors</u> to get another)

any element of V will be called <u>a vector</u>

- Scalar multiplication: $$\mathbb{R} \times V \longrightarrow V$$ $$(\alpha, \vec{v}) \longmapsto \alpha \vec{v}$$

These two operations are required to satify the following list of properties: (see Lecture 13 page 2).

<u>Addition</u>:

(A1) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ for every $\vec{u}, \vec{v}$ in V.

(A2) $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$ for every $\vec{u}, \vec{v}, \vec{w}$ in V.

(A3) There is <u>an element $\vec{0}$ in V</u> for which (neutral element)
$$\vec{0} + \vec{v} = \vec{v} \quad (= \vec{v} + \vec{0}).$$

(A4) Given $\vec{v}$ in V, we can find "$\underbrace{-\vec{v}}_{\text{(additive inverse)}}$" so that
$$\vec{v} + (-\vec{v}) = \vec{0}.$$

<u>Scalar Multiplication</u>:

(M1) $a(b\vec{v}) = (ab)\vec{v}$ for every $a, b \in \mathbb{R}$, $\vec{v} \in V$.

$$\left.\begin{array}{ll} \text{(M2)} & a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v} \\ \text{(M3)} & (a+b)\vec{u} = a\vec{u} + b\vec{u} \end{array}\right\} \text{for every } a, b \in \mathbb{R} \text{ and } \vec{u}, \vec{v} \in V.$$

(M4) $1 \cdot \vec{u} = \vec{u}$.

Old examples

- $\mathbb{R}^N$ is a vector space (Lecture 13, page 1)

- for any m, n positive integers;

$$M_{m\times n}(\mathbb{R}) = \text{set of } \underline{\text{all}} \ m\times n \text{ matrices}$$

is a vector space (Lecture 6, page 1).

- Null space and range of a matrix are vector spaces. (Lecture 14, pages 3, 4).

Some new examples

(i) $\mathbb{F}$ = set of all functions $\mathbb{R} \to \mathbb{R}$.
(of one variable)

(ii) $C([0,1])$ = all continuous functions defined on the interval $[0,1]$.

(iii) P_n = all polynomials of degree at most n.

$$= \left\{ a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n \ \middle|\ \begin{array}{c} a_0, a_1, \dots, a_n \in \mathbb{R} \\ \text{arbitrary} \end{array} \right\}$$

Addition: $(a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n) + (b_0 + b_1 x + b_2 x^2 + \dots + b_n x^n)$

$$= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \dots + (a_n + b_n)x^n.$$

Scalar Multiplication: $c \cdot (a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n)$

$$= (ca_0) + (ca_1)x + (ca_2)x^2 + \dots + (ca_n)x^n.$$

Neutral element   Zero polynomial $0 + 0x + 0x^2 + \dots + 0x^n$.

(21.2) Some useful properties of vector spaces.

Let V be a vector space.

(i) Cancellation.   If $\vec{u} + \vec{v} = \vec{u} + \vec{w}$ then $\vec{v} = \vec{w}$.

(Reason: Add $(-\vec{u})$ to both sides of the equation $\vec{u} + \vec{v} = \vec{u} + \vec{w}$.)

(ii) Neutral element is unique. Meaning: if there are two elements $\vec{0}$ and $\vec{0}'$ of V for which
$$\left.\begin{aligned} \vec{0} + \vec{v} &= \vec{v} \\ \vec{0}' + \vec{v} &= \vec{v} \end{aligned}\right\} \text{for every } \vec{v} \text{ in } V$$

then $\vec{0} = \vec{0}'$.

(Proof.- $\vec{0} = \vec{0}' + \vec{0} = \vec{0} + \vec{0}' = \vec{0}'$.)

(iii) Additive inverse is unique. Meaning: given $\vec{v}$ in V, if there are two elements $\vec{u}_1$ and $\vec{u}_2$ such that
$$\begin{aligned} \vec{u}_1 + \vec{v} &= \vec{0} \text{ and} \\ \vec{u}_2 + \vec{v} &= \vec{0} \end{aligned}$$

then $\vec{u}_1 = \vec{u}_2$.

(Proof.-
$$\begin{aligned} \vec{u}_1 &= \vec{u}_1 + \vec{0} = \vec{u}_1 + \vec{u}_2 + \vec{v} = \vec{u}_1 + \vec{v} + \vec{u}_2 \\ &= \vec{0} + \vec{u}_2 = \vec{u}_2. \end{aligned}$$
)

(iv) $0 \cdot \vec{v} = \vec{0}$ for any $\vec{v}$ in V.

(Proof $0 \cdot \vec{v} = (0 + 0) \cdot \vec{v} = (0 \cdot \vec{v}) + (0 \cdot \vec{v})$. So $\vec{0} + (0 \cdot \vec{v}) = (0 \cdot \vec{v}) + (0 \cdot \vec{v})$
by (i) - cancellation - we get $(0 \cdot \vec{v}) = \vec{0}$)

(v) $a \cdot \vec{0} = \vec{0}$ for any $a \in \mathbb{R}$.

(Proof : $a \cdot \vec{0} = a \cdot (\vec{0} + \vec{0}) = a \cdot \vec{0} + a \cdot \vec{0}$

So $\vec{0} + (a \cdot \vec{0}) = (a \cdot \vec{0}) + (a \cdot \vec{0})$.

By cancellation : $a \cdot \vec{0} = \vec{0}$.)

(21.3) <u>Subspaces</u>. Let V be a vector space. A <u>subset $W \subset V$</u> is called <u>a subspace</u> if (i) $\vec{0}$ is in W.

(ii) $\vec{v}_1, \vec{v}_2$ in W implies $\vec{v}_1 + \vec{v}_2$ is in W.

(iii) $\vec{v}$ in W and $c \in \mathbb{R}$ implies $c \cdot \vec{v}$ is in W.

Old examples. – • A : $m \times n$ matrix $\rightsquigarrow$ Null space $N(A) \subset \mathbb{R}^n$ is a subspace.

Range of $A = R(A) \subset \mathbb{R}^m$ is a subspace

New examples. – • $\mathbb{F}$ = functions of one variable

$\mathcal{S} = \{ f(x) \text{ such that } f'' = -f \} \subset \mathbb{F}$ is a subspace

P_n = polynomials of degree atmost n

$= \{ a_0 + a_1 x + \ldots + a_n x^n \mid a_0, a_1, \ldots, a_n \in \mathbb{R} \text{ arbitrary} \} \subset \mathbb{F}$ is a subspace

Non- ~~ex~~ examples :

- $S = \left\{ f(x) \;\middle|\; \int_0^1 f(x)\,dx = 1 \right\} \subset \mathbb{E}$ is NOT a subspace

(Why? . Zero function is not in this set.

(not necessary) $\longrightarrow$. f, g in S means $\int_0^1 f(x)dx = 1$ and $\int_0^1 g(x)dx = 1$

but then $\int_0^1 (f(x)+g(x))\,dx = 2.$)

- $S = \{$ polynomials $a_0 + a_1 x + \dots + a_n x^n = p(x)$ in $\mathcal{P}_n$ such that $p(0) = 1 \}$

$\subset \mathcal{P}_n$ is NOT a subspace (Zero polynomial is NOT in S).

- Recall $M_{m\times n}(\mathbb{R})$ = set of all $m\times n$ matrices, is a vector space.

$\rightarrow$ $S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ such that } ad-bc \neq 0 \right\} \subset M_{2\times 2}(\mathbb{R})$ is NOT a subspace (again $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin S$).

$\rightarrow$ $X = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ such that } ad-bc = 0 \right\} \subset M_{2\times 2}(\mathbb{R})$ is also NOT a subspace

Reason $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are in X. But their sum

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is NOT in X.