# Lecture 22

(22.0) Recall - last time we defined a vector space (over real numbers) as: a vector space $V$ is a set on which 2 operations are given:

- addition. For any $\vec{u}, \vec{v}$ in $V$, we have a new element $\vec{u}+\vec{v} \in V$.
- scalar multiplication. For any $c \in \mathbb{R}$ and $\vec{v} \in V$, we have an element $c.\vec{v} \in V$.

These operations must satisfy 8 properties listed last time:

Addition: (A1) Addition is commutative. Meaning
$$\vec{u}+\vec{v} = \vec{v}+\vec{u}$$

(A2) Addition is associative. Meaning $\vec{u}+(\vec{v}+\vec{w}) = (\vec{u}+\vec{v})+\vec{w}$

(A3) There exists a neutral element. Meaning - there is a vector $\vec{0}$ in $V$ for which $\vec{0}+\vec{v}=\vec{v}$ for any $\vec{v}$ in $V$.

(A4) Additive inverses exist. For any $\vec{v}$ in $V$, there is some $\vec{u}$ in $V$ so that $\vec{u}+\vec{v}=\vec{0}$.

(Remark: We proved that neutral element is unique, so we can say "$\vec{0}$ is the neutral element of $V$" or just "the zero vector".
We also showed that additive inverse is unique - so now we can say "$-\vec{v}$ is the additive inverse of $\vec{v}$".)

Scalar multiplication (M1) Associative – Meaning:

$$a(b\vec{v}) = (ab)\vec{v}.$$

(M2) Multiplication distributes over addition Meaning:

(& M3)

$$(a+b)\vec{v} = a\vec{v} + b\vec{v}$$

$$a(\vec{u}+\vec{v}) = a\vec{u} + a\vec{v}$$

(M4) $1 \in \mathbb{R}$ is neutral for multiplication : $1.\vec{v} = \vec{v}$ for every $\vec{v}$ in $V$.

- Elements of $V$ will be referred to as vectors, and elements of $\mathbb{R}$ as scalars (as before).

(22.1) We also defined subspaces. Recall that a subset $W \subset V$ is a subspace if (i) $\vec{0} \in W$ (Zero vector of $V$ lies in $W$)

(ii) $W$ is closed under addition :

$\vec{u}, \vec{v} \in W$ implies $\vec{u}+\vec{v} \in W$.

(iii) $W$ is closed under scalar multiplication :

$c \in \mathbb{R}$, $\vec{u} \in W$ implies $c.\vec{u} \in W$.

(Note: as before $W = \{\vec{0}\} \subset V$ are smallest subspaces.)
$W = V \subset V$ largest

(22.2) Some new examples, and notations. (from Lecture 21).

- $M_{m\times n}(\mathbb{R})$ = set of all $m\times n$ matrices.
- $\mathbb{E}$ = set of all functions of one variable.
- $\mathbb{P}_n$ = set of all polynomials of degree at most $n$.
- $C([0,1])$ = set of all continuous functions defined on the interval $\{0 \leq x \leq 1\}$.

These are all vector spaces. Note that $\mathbb{P}_n \subset \mathbb{E}$ is a subspace. Also, $\mathbb{P}_n \subset C([0,1])$ is a subspace.

Example. Let $n \geq 2$ and $M_{n\times n}(\mathbb{R})$ be the vector space of all (square) $n\times n$ matrices.

$$\{A \in M_{n\times n}(\mathbb{R}) \text{ such that } \boxed{A^T = A}\} \subset M_{n\times n}(\mathbb{R})$$

is a subspace.

such matrices are called symmetric

Reason: $O_{n\times n}$ = $n\times n$ matrix with all entries $= 0$.

(i) $O_{n\times n}^T = O_{n\times n}$ ✓

(ii) $A^T = A$ and $B^T = B$ implies $(A+B)^T = A^T + B^T = A+B$ ✓

(iii) $c \in \mathbb{R}$; $A \in M_{n\times n}(\mathbb{R})$ such that $A^T = A$ implies $(cA)^T = cA^T = cA$. ✓

In general the <u>intuitive idea</u> is that "subspaces are defined via linear homogeneous equations".

e.g.: $X = \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{R}) \text{ such that } \underbrace{ab = 0}_{\text{NOT linear!}} \right\} \subset M_{2\times 2}(\mathbb{R})$

is not a subspace.

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ are in $X$ but their sum $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ is NOT.

- $Y = \left\{ p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \text{ such that } p''(0) = \underbrace{5}_{\text{not homogeneous}} \right\} \subset \mathcal{P}_3$

is not a subspace

Zero polynomial $\notin Y$.

(22.3) <u>Spanning sets</u>. Let $V$ be a vector space.

Let $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\} \subset V$ be a set of vectors.

$\boxed{\text{Span}(S)}$ is defined (as before) as the set of all possible linear combinations of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$.

$$\text{Span}(S) = \{ a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_p\vec{v}_p \text{ where } a_1, a_2, \dots, a_p \in \mathbb{R} \text{ are arbitrary} \}$$

$\subset V$.

Just as before, we have the result that Span(S) is a subspace of $V$.

( Proof. - (i) $\vec{0} = 0 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2 + \dots + 0 \cdot \vec{v}_p \in \text{Span}(S)$ ✓

(ii) $\left.\begin{array}{l} a_1 \vec{v}_1 + a_2 \vec{v}_2 + \dots + a_p \vec{v}_p \\ b_1 \vec{v}_1 + b_2 \vec{v}_2 + \dots + b_p \vec{v}_p \end{array}\right\}$ two elements of Span(S)

implies that their sum $= (a_1+b_1)\vec{v}_1 + (a_2+b_2)\vec{v}_2 + \dots + (a_p+b_p)\vec{v}_p$ is also in Span(S). ✓

(iii) $c \in \mathbb{R}$ and $a_1\vec{v}_1 + \dots + a_p\vec{v}_p \in \text{Span}(S)$ implies

$c(a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_p\vec{v}_p) = (ca_1)\vec{v}_1 + (ca_2)\vec{v}_2 + \dots + (ca_p)\vec{v}_p \in \text{Span}(S)$ ✓)

Definition. Let $W \subset V$ be a subspace, and $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$ a finite set of vectors in $W$.

If $\boxed{\text{Span}(S) = W}$, we say S is a spanning set for W.

(22.4) Linear independence. Again let $V$ be a vector space and $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\} \subset V$ a set of vectors from $V$.

We say that $S$ is a linearly dependent set of vectors, or $\vec{v}_1, \dots, \vec{v}_p$ are linearly dependent if there exist scalars $a_1, a_2, \dots, a_p$; not all zero, such that

$$\boxed{a_1 \vec{v}_1 + a_2 \vec{v}_2 + \dots + a_p \vec{v}_p = \vec{0}}$$

As before, such an equation is referred to as a "dependence relation among $\vec{v}_1, \dots, \vec{v}_p$".

A set of vectors is linearly independent if it is not linearly dependent. Meaning: $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$ is linearly independent if:

$$a_1 \vec{v}_1 + a_2 \vec{v}_2 + \dots + a_p \vec{v}_p = \vec{0} \text{ implies } a_1 = a_2 = \dots = a_p = 0.$$

(22.5) A set of vectors $B = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\} \subset V$ is called a basis of $V$ if

- $B$ is a spanning set for $V$. That is, $\mathrm{Span}(B) = V$.
- $B$ is linearly independent.

Examples. - (1) $\mathcal{P}_n$ = set of all polynomials of degree at most $n$.

$B = \{1, x, x^2, \dots, x^n\} \subset \mathcal{P}_n$ is a basis.

- $p(x) \in \mathcal{P}_n$ means $p(x) = a_0 \cdot 1 + a_1 \cdot x + \dots + a_n \cdot x^n$
$\in \text{Span}(\{1, x, x^2, \dots, x^n\})$.

- If $b_0, b_1, \dots, b_n \in \mathbb{R}$ are such that

$$b(x) = b_0 + b_1 x + \dots + b_n x^n = 0$$

then
$$0 = b(0) = b_0.$$
$$0 = b'(0) = b_1. \qquad \text{implies } b_0 = b_1 = \dots = b_n = 0$$
$$0 = b''(0) = 2b_2.$$
$$0 = b'''(0) = 6 \cdot b_3$$
$$\vdots$$

Hence $\{1, x, x^2, \dots, x^n\}$ is linearly independent.

(2) Show that $\{\sin(x), \cos(x)\}$ is a linearly independent set.

Sol.: If $a, b \in \mathbb{R}$ are such that $\boxed{a\sin(x) + b\cos(x) = 0}$

then: setting $x = 0$ gives $a \underbrace{\sin(0)}_{=0} + b \underbrace{\cos(0)}_{=1} = 0$

i.e. $b = 0$.

Setting $x = \frac{\pi}{2}$ gives $a \underbrace{\sin\left(\frac{\pi}{2}\right)}_{=1} + b \underbrace{\cos\left(\frac{\pi}{2}\right)}_{=0} = 0$ i.e. $a = 0$.

(3) Let $W$ = set of all $3\times3$ symmetric matrices $\subset M_{3\times3}(\mathbb{R})$

Find a basis of $W$.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$ is in $W$ means $A = A^T$, i.e.,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

That is, $a_{12} = a_{21}$ ; $a_{23} = a_{32}$ and $a_{13} = a_{31}$.

Typical element of $W$:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = a_{11}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_{12}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(*) \qquad + a_{13}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + a_{22}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$+ a_{23}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + a_{33}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

So, if $S = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right\}$

then $\mathrm{Span}(S) = W$. $(*)$ also implies that $S$ is linearly independent (only numbers $a_{11}, a_{22}, a_{33}, a_{12}, a_{13}, a_{23}$ which make R.H.S. of $(*) = O_{3\times3}$ ; are all 0's).

(hence a basis of $W$).