Lecture 23

(23.0) Recall that last time we defined spanning sets, linearly independent set of vectors and bases.

Let V be a vector space. Given a set of vectors

$$S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\} \subset V$$

we defined:

- $\text{Span}(S) = \{a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + a_p\vec{v}_p$ where $a_1, a_2, \ldots, a_p \in \mathbb{R}$ are arbitrary$\} \subset V$.

  Span(S) is a subspace of V.

- We say S is a linearly independent set of vectors if $a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + a_p\vec{v}_p = \vec{0}$ implies $a_1 = a_2 = \ldots = a_p = 0$.

Definition: Let $W \subset V$ be a subspace and

$$B = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\} \subset W \quad \text{(subset)}.$$

B is a basis of W if (i) $\text{Span}(B) = W$

(ii) B is linearly independent.

(23.1) Some more examples.

(i) Let $V = M_{2\times 3}(\mathbb{R})$ = set of all 2×3 matrices.

$$B = \left\{ \begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&1\end{bmatrix} \right\} \subset V$$

is a basis of V.

$$\begin{bmatrix}a&b&c\\d&e&f\end{bmatrix} = a\underbrace{\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}}_{E_{11}} + b\underbrace{\begin{bmatrix}0&1&0\\0&0&0\end{bmatrix}}_{E_{12}} + c\underbrace{\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}}_{E_{13}} \quad - (*)$$

(typical 2x3 matrix)

$$+ d\underbrace{\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}}_{E_{21}} + e\underbrace{\begin{bmatrix}0&0&0\\0&1&0\end{bmatrix}}_{E_{22}} + f\underbrace{\begin{bmatrix}0&0&0\\0&0&1\end{bmatrix}}_{E_{23}}$$

- So, $\mathrm{Span}(B) = V$.
- if $a, b, c, d, e, f \in \mathbb{R}$ are such that

$$a E_{11} + b E_{12} + c E_{13} + d E_{21} + e E_{22} + f E_{23} = \begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}$$

then (from the left-hand side of $(*)$), $a = b = c = d = e = f = 0$.

This means that B is a linearly independent set.

(ii) Again, let $V = M_{3\times3}(\mathbb{R})$ = vector space of all 3×3 matrices.

$$W = \{ A : 3\text{x}3 \text{ matrix such that } \underline{A^T = -A} \} \subset V$$

such matrices are called skew-symmetric

A typical element of W:

$$\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix} = \begin{bmatrix}-a_{11}&-a_{21}&-a_{31}\\-a_{12}&-a_{22}&-a_{32}\\-a_{13}&-a_{23}&-a_{33}\end{bmatrix}$$

i.e.

$a_{11} = a_{22} = a_{33} = 0$.

$a_{12} = -a_{21}$

$a_{13} = -a_{31}$

$a_{23} = -a_{32}$.

So every 3×3 skew-symmetric matrix has the following form:

$$\begin{bmatrix} 0 & a & c \\ -a & 0 & b \\ -c & -b & 0 \end{bmatrix} = a \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

$$\mathcal{B} = \left\{ \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \right\} \subset W \text{ is a basis.}$$

(23.2) Not every vector space has a finite basis. For example,

$V = \mathbb{F}$ = functions of one variable.

We already saw last time that $\{1, x, x^2, x^3, \dots, x^n\}$ is a linearly independent set, for any $n \geq 0$. So, we can find arbitrarily large linearly independent sets in V. This means V can not possibly have a finite basis.

→ In this course we will mostly focus on vector spaces that do admit a finite basis.

(23.3) Properties of bases.

Let V be a vector space and let

$$B = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\} \subset V \quad \text{(subset)}$$

be a basis of V.

1. For every $\vec{v} \in V$, there are uniquely determined scalars $a_1, a_2, \dots, a_p \in \mathbb{R}$ so that

$$\vec{v} = a_1 \vec{v}_1 + a_2 \vec{v}_2 + \dots + a_p \vec{v}_p$$

(Proof. Since $\text{Span}(B) = V$, we can express $\vec{v}$ as a linear combination of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$. Assume that we ~~many~~ have two such expressions:

$$\begin{aligned} \vec{v} &= a_1 \vec{v}_1 + a_2 \vec{v}_2 + \dots + a_p \vec{v}_p \\ &= b_1 \vec{v}_1 + b_2 \vec{v}_2 + \dots + b_p \vec{v}_p \end{aligned}$$

. Taking their difference, we get:

$$(a_1 - b_1)\vec{v}_1 + (a_2 - b_2)\vec{v}_2 + \dots + (a_p - b_p)\vec{v}_p = \vec{v} - \vec{v} = \vec{0}.$$

As $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$ is a linearly independent set, we must have

$$\begin{aligned} a_1 - b_1 &= 0 \\ a_2 - b_2 &= 0 \\ &\vdots \\ a_p - b_p &= 0 \end{aligned}$$

That is,

$$\begin{aligned} a_1 &= b_1 \\ a_2 &= b_2 \\ &\vdots \\ a_p &= b_p \end{aligned}$$

$\square$)

2. If $S = \{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_q\} \subset V$ is any set of q vectors, <u>with $q > p$</u>, then S is <u>linearly dependent</u>.

(Proof. Since B is a basis, there are unique expressions for $\vec{w}$'s as linear combinations of $\{\vec{v}_1, \dots, \vec{v}_p\}$.

$$\vec{w}_1 = a_{11}\vec{v}_1 + a_{21}\vec{v}_2 + \dots + a_{p1}\vec{v}_p$$
$$\vec{w}_2 = a_{12}\vec{v}_1 + a_{22}\vec{v}_2 + \dots + a_{p2}\vec{v}_p \rightsquigarrow A = (a_{ij})$$
$p \times q$ size matrix.
$$\vdots$$
$$\vec{w}_q = a_{1q}\vec{v}_1 + a_{2q}\vec{v}_2 + \dots + a_{pq}\vec{v}_p$$

A linear expression $x_1\vec{w}_1 + x_2\vec{w}_2 + \dots + x_q\vec{w}_q = \vec{0}$ is same as a homogeneous system

$$\underbrace{A}_{p\times q} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_q \end{bmatrix}}_{q \times 1} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{p \times 1}.$$

$\#$ equations $= p$
$< q = \#$ unknowns

Hence, non-trivial solutions exist.

This proves that $\{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_q\}$ has to be linearly dependent.)

3. If $\tilde{B} = \{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_q\} \subset V$ is another basis of V, then $p = q$.

(Proof. If $q > p$, then $\tilde{B}$ would be linearly dependent, but it is a basis (in particular linearly independent)).

This means $q \leq p$. Arguing the same way, after switching the roles of B and $\tilde{B}$, we conclude that $p \leq q$. Hence, $p = q$.)

(23.4) <u>Consequences of these properties.</u>

1. <u>Dimension of V</u>, <u>$\dim(V)$</u>, is defined to be <u>number of vectors in a basis.</u> This is a well-defined non-negative integer (assuming V admits a finite basis) because of property 3 on page 5 above.

2. <u>Coordinates relative to a basis</u>. Assume $\dim(V) = p$.

   Let $B = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ be a basis of V.

To each $\vec{v} \in V$, we assign scalars $a_1, a_2, \ldots, a_p \in \mathbb{R}$ according to: $\vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + a_p\vec{v}_p$ (see Property 1 on page 4 above).

$$\underset{(\text{in } V)}{\vec{v}} \rightsquigarrow \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix} \quad (\text{in } \mathbb{R}^p).$$

(this assigment depends on the choice of a basis)

We say $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix}$ are <u>coordinates of $\vec{v}$</u> <u>relative to basis B</u>

$$\underbrace{B = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\} \text{ a basis of } V}_{\text{(Given)}}$$

$[\vec{v}]_B := \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix}$ in $\mathbb{R}^p$, uniquely determined numbers so that

$$\vec{v} = \underbrace{a_1}\vec{v}_1 + \underbrace{a_2}\vec{v}_2 + \dots + \underbrace{a_p}\vec{v}_p$$

coordinates of $\vec{v}$ relative to B. (basis)

e.g. (a) When $V = \mathbb{R}^p$ with the standard basis $B = \{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_p\}$; coordinates are the usual "entries" of a vector.

<u>Warning</u>: Ordering of basis vectors is important. For instance for $B = \left\{ \vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \subset \mathbb{R}^3$ <u>basis</u>.

Coordinates of an element

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a \cdot \vec{v}_2 + b \cdot \vec{v}_1 + c \cdot \vec{v}_3 \rightsquigarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix}_B = \begin{bmatrix} b \\ a \\ c \end{bmatrix}$$

(b) Let $V \subset M_{2\times 2}(\mathbb{R})$ be given by

$$\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a + d = 0 \right\}.$$

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\} \subset V$$

is a basis. Since a typical element of V has the form

$$\begin{bmatrix} a & b \\ c & -a \end{bmatrix} = a \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

So coordinates relative to $\mathcal{B}$ allows us to see V as $\mathbb{R}^3$

$$\begin{bmatrix} a & b \\ c & -a \end{bmatrix} \rightsquigarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \left(\begin{bmatrix} a & b \\ c & -a \end{bmatrix} \right)_{\mathcal{B}}$$

coordinates relative to $\mathcal{B}$.

We often view this as a "linear transformation"

$$\begin{array}{ccc} V & \longrightarrow & \mathbb{R}^p \\ \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ \vec{v} & \rightsquigarrow & [\vec{v}]_{\mathcal{B}} \end{array}$$

(to be defined again when we will verify that this is indeed a linear transformation.)