Lecture 24

(24.0) Recall : last time we defined "<u>coordinates relative to a basis</u>". If V is a vector space and

$$B = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\} \subset V \quad \underline{\text{is a basis of } V},$$
(subset)

(Meaning - every $\vec{v}$ in V can be written <u>uniquely</u> as a linear combination : $\vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + a_p\vec{v}_p$.)

then $[\vec{v}]_B$ = coordinates of $\vec{v}$ relative to the basis B

(for $\vec{v}$ in V) $= \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix} \in \mathbb{R}^p$ are <u>defined via the unique expression</u> of $\vec{v}$ <u>as a linear combination</u> of $\vec{v}_1, \ldots, \vec{v}_p$:

$$\vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + a_p\vec{v}_p.$$

(24.1) We revisited the definition of dimension last time.

$\dim(V)$ = dimension of V = number of vectors in a (finite) basis of V

(assuming V has a finite basis).

- If V is a vector space which does not have a finite basis (e.g. $V = \mathbb{E}$ = vector space of all functions of one variable) we say V is <u>infinite-dimensional</u>. As mentioned in the last lecture, we will mainly talk about finite-dimensional vector spaces only.

- If $V = \{\vec{0}\}$ (vector space containing only one vector - namely the zero vector)

  then we agreed to take $B = \phi$ empty set as its basis so as to have $\dim(\{\vec{0}\}) = 0$ = number of elements in the empty set.

(24.2) Example. Let $V = M_{2\times 2}(\mathbb{R})$ and let $B \subset V$ consist of the following matrices.

$$B = \left\{ A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \cancel{A_4} \right.$$
$$\left. A_4 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}$$

Verify that B is a basis of V and write the coordinates of $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ relative to B.

<u>Sol.</u> A typical element of V is a 2×2 matrix, say $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We want to know how to write it as a linear combination of A_1, A_2, A_3 and A_4.

i.e. given $a, b, c, d \in \mathbb{R}$, find x, y, z, w such that

$$x \cdot A_1 + y \cdot A_2 + z \cdot A_3 + w \cdot A_4 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \quad \text{— } (*)$$

Left-hand side of this equation is :

$$\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} y & 0 \\ y & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & z \end{bmatrix} + \begin{bmatrix} 0 & w \\ 0 & w \end{bmatrix}$$

$$= \begin{bmatrix} x+y & w \\ y & z+w \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ (right-hand side)}$$

which gives 4 eq$^{\underline{n}}$s in 4 unknowns :

$$\left.\begin{aligned} x+y &= a \\ y &= c \\ w &= b \\ z+w &= d \end{aligned}\right\} \Rightarrow \begin{aligned} x &= a-c \\ y &= c \\ w &= b \\ z &= d-b \end{aligned}$$

Since for any $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in V$, the equation $(*)$ has a unique solution, we conclude that $\{A_1, A_2, A_3, A_4\}$ is a basis of V.

Also, when $\begin{aligned} a &= 2 \\ b &= -1 \\ c &= -1 \\ d &= 2 \end{aligned}$, we get $\begin{aligned} x &= a-c = 2-(-1) = 3 \\ y &= c = -1 \\ w &= b = -1 \\ z &= d-b = 2-(-1) = 3 \end{aligned}$

Meaning $$\left(\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\right)_B = \begin{bmatrix} 3 \\ -1 \\ -1 \\ 3 \end{bmatrix} \in \mathbb{R}^4.$$

Coordinates of $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ relative to B.

$\square$

(24.3) Linear Transformations

Let V and W be two vector spaces. A <u>linear transformation</u> from V to W is a function:

$$T : V \longrightarrow W$$

(T takes inputs from V and gives outputs = elements of W)

such that (i) $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2)$

for any two vectors $\vec{v}_1$ and $\vec{v}_2$ from V.

(ii) $T(c \cdot \vec{v}) = c \cdot T(\vec{v})$

for any scalar $c \in \mathbb{R}$ and vector $\vec{v} \in V$.

(in plain english: T respects <u>addition</u> and <u>scalar multiplications</u>, two things that define a vector space)

Remark.- As before it follows from the definition that a linear transformation must send zero vector to zero vector. Since there are two vector spaces involved here, let us denote by $\vec{0}_V$ the zero vector of V, and by $\vec{0}_W$ the zero vector of W. Then, for $T : V \to W$, a linear transformation,

$$T(\vec{0}_V) = \vec{0}_W .$$

(Proof - Take $c = 0$ in (ii)).

(24.4) Some examples of linear transformations

(i) Let $V = P_3$ = set of all polynomials of degree ≤ 3.

$W = P_2$. $T = \frac{d}{dx} : V \to W$ is a linear transformation, by the usual properties of derivatives:

$$(f+g)' = f' + g'$$

$$(cf)' = cf' \qquad (c \in \mathbb{R})$$

Explicitly $T(a_0 + a_1 x + a_2 x^2 + a_3 x^3) = a_1 + 2a_2 x + 3a_3 x^2$.

(ii) Again $V = P_3$. Let $T : V \to V$ be given by

$$(Tf)(x) = f(x+1).$$

Check: T is a linear transformation.

$$(T(f+g))(x) = (f+g)(x+1) = f(x+1) + g(x+1)$$
$$= (T(f) + T(g))(x). \checkmark$$

Explicitly, $T(a_0 + a_1 x + a_2 x^2 + a_3 x^3)$

$$= a_0 + a_1(x+1) + a_2(x+1)^2 + a_3(x+1)^3$$
$$= a_0 + a_1(x+1) + a_2(x^2 + 2x + 1) + a_3(x^3 + 3x^2 + 3x + 1)$$
$$= (a_0 + a_1 + a_2 + a_3) + (a_1 + 2a_2 + 3a_3)x$$
$$+ (a_2 + 3a_3)x^2 + a_3 x^3.$$

(iii) $V = C([0,1])$ continuous function on the closed interval $0 \le x \le 1$.

$T : V \longrightarrow \mathbb{R}$, $T(f) = \int_0^1 f(x)\,dx$ (= area under the graph of f) is a linear transformation.

(iv) $V = \mathbb{R}^n$, $W = \mathbb{R}^m$ (old example)

Given A an $m \times n$ size matrix, we get a linear transformation $V \longrightarrow W$.

$$\vec{x} \longmapsto A\vec{x}$$

(24.5) <u>Null space and range of a linear transformation</u>:

Let V and W be two vector spaces, and let $T : V \to W$ be a linear transformation

<u>Null Space of T</u> , $\mathcal{N}(T) \subset V$, is defined (as before):

$$\mathcal{N}(T) = \{\vec{v} \in V \text{ such that } T(\vec{v}) = \vec{0}_W\} \subset V.$$

It is <u>a subspace of V</u> by the same proof as before:

(i) $\vec{v}_1, \vec{v}_2 \in \mathcal{N}(T)$ means $T(\vec{v}_1) = \vec{0}_W$ and $T(\vec{v}_2) \in \vec{0}_W$.

Then $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2) = \vec{0}_W + \vec{0}_W = \vec{0}_W$. ✓

(ii) If $\vec{v} \in \mathcal{N}(T)$ and $c \in \mathbb{R}$, then

$T(c\vec{v}) = cT(\vec{v}) = c \cdot \vec{0}_W = \vec{0}_W$. ✓

Range of T, $R(T) \subset W$ is defined as

$$R(T) = \left\{ \vec{w} \in W \text{ such that } \vec{w} = T(\vec{v}) \text{ for some } \vec{v} \in V \right\} \subset W.$$

Again $R(T)$ is a subspace of W.

As before, we define nullity and rank of T:

$$\boxed{\begin{aligned} \text{Nullity}(T) &= \dim(\mathcal{N}(T)) \\ \text{Rank}(T) &= \dim(R(T)) \end{aligned}}$$

(24.6) Definition. Let $T : V \to W$ be a linear transformation between two vector spaces.

We say T is one-to-one if $\boxed{T(\vec{v}_1) = T(\vec{v}_2) \text{ implies } \vec{v}_1 = \vec{v}_2}$.

We say T is onto if $\boxed{\text{every element of } W \text{ comes from some element of } V}$

(i.e., T is onto when $R(T) = W$. Still, in other words, T is onto when $\text{rank}(T) = \dim(W)$.)

T is one-to-one if and only if $N(T) = \{\vec{0}_V\}$.

(i.e. $\text{Nullity}(T) = 0$).

(Proof. Assume T is one-to-one. If $\vec{v} \in N(T)$, then

$$T(\vec{v}) = \vec{0}_W = T(\vec{0}_V).$$

by definition of $N(T)$

as T is a linear transformation $T(\vec{0}_V) = \vec{0}_W$

So $T(\vec{v}) = T(\vec{0}_V)$, and by definition of one-to-one, we must have $\vec{v} = \vec{0}_V$. We have proved that every element of $N(T)$ is $\vec{0}_V$, i.e., $N(T) = \{\vec{0}_V\}$.

Now assume that $N(T) = \{\vec{0}_V\}$. Let us prove that T is one-to-one. If $\vec{v}_1$ and $\vec{v}_2$ are two vectors in V so that $\boxed{T(\vec{v}_1) = T(\vec{v}_2)}$, then $T(\vec{v}_1 - \vec{v}_2) = T(\vec{v}_1) - T(\vec{v}_2)$

$$= \vec{0}_W.$$

This implies $\vec{v}_1 - \vec{v}_2 \in N(T)$. But $N(T) = \{\vec{0}_V\}$, so

$$\vec{v}_1 - \vec{v}_2 = \vec{0}_V \Rightarrow \boxed{\vec{v}_1 = \vec{v}_2}. \qquad \square\ )$$