(25.0) Recall – last time we defined linear transformations and related concepts.

- A <u>linear transformation</u> from a vector space $V$ to a vector space $W$ is a function $T : V \to W$ such that

  (i) $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2)$ for any two vectors $\vec{v}_1, \vec{v}_2$ in $V$.

  (ii) $T(c\vec{v}) = c \cdot T(\vec{v})$ for any vector $\vec{v}$ in $V$ and scalar $c \in \mathbb{R}$.

- We say $T$ is <u>one-to-one</u> if $\boxed{\vec{v}_1 \neq \vec{v}_2 \text{ implies } T(\vec{v}_1) \neq T(\vec{v}_2)}$

  and $T$ is <u>onto</u> if $\boxed{\text{for every } \vec{w} \in W, \text{ there is some } \vec{v} \in V \text{ such that } T(\vec{v}) = \vec{w}}$.

- <u>Null space of $T$</u>, $N(T) = \{\vec{v} \text{ in } V \text{ such that } T(\vec{v}) = \vec{0}_W\} \subset V$ (subspace)

  <u>Nullity $(T)$</u> = dimension of $N(T)$

- <u>Range of $T$</u>, $R(T) = \{\vec{w} \text{ in } W \text{ such that } \vec{w} = T(\vec{v}) \text{ for some } \vec{v} \text{ in } V\} \subset W$ (subspace)

  <u>Rank $(T)$</u> = dimension of $R(T)$

We proved that

- $T$ is <u>one-to-one</u> if and only if <u>$N(T) = \{\vec{0}_V\}$</u> (i.e. Nullity $(T) = 0$).
- $T$ is <u>onto</u> if and only if <u>$R(T) = W$</u> (i.e. Rank $(T) = \dim(W)$).

<u>Notation</u>:
$$\left.\begin{array}{l} \vec{0}_V \in V \\ \vec{0}_W \in W \end{array}\right\} \text{ denote } \underline{\text{the zero vectors}} \text{ of } V \text{ and } W.$$

(25.1) Example of coordinates relative to a basis. as a linear transformation

Let $V$ be a vector space of dimension $p$.

Let $B = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\} \subset V$ be a basis of $V$.

Let $T : V \longrightarrow \mathbb{R}^p$ be defined by

$$T(\vec{v}) = [\vec{v}]_B = \text{coordinates of } \vec{v} \text{ relative to the basis } B.$$

<u>$T$ is a linear transformation</u>. It is <u>both one-to-one and onto</u>.

Let us check that $T$ is a linear transformation. Recall that

$$[\vec{v}]_B = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix} \quad \text{means } \vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_p\vec{v}_p.$$

(i) If $[\vec{v}]_B = \begin{bmatrix} a_1 \\ \vdots \\ a_p \end{bmatrix}$ and $[\vec{u}]_B = \begin{bmatrix} b_1 \\ \vdots \\ b_p \end{bmatrix}$, then

$$\begin{aligned} \vec{v} &= a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_p\vec{v}_p \\ \vec{u} &= b_1\vec{v}_1 + b_2\vec{v}_2 + \dots + b_p\vec{v}_p \end{aligned} \rightsquigarrow \vec{v} + \vec{u} = (a_1+b_1)\vec{v}_1 + (a_2+b_2)\vec{v}_2 + \dots + (a_p+b_p)\vec{v}_p$$

Meaning $[\vec{v} + \vec{u}]_B = [\vec{v}]_B + [\vec{u}]_B$ ✓

(ii) If $[\vec{v}]_B = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix}$ then $\vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_p\vec{v}_p$

$\rightsquigarrow c\vec{v} = (ca_1)\vec{v}_1 + (ca_2)\vec{v}_2 + \dots + (ca_p)\vec{v}_p$

Meaning $[c\vec{v}]_B = c[\vec{v}]_B$ ✓

Now let us check that $T$ is one-to-one. Recall that for this we only have to verify that $N(T) = \{\vec{0}_V\}$.

If $\vec{v}$ in $V$ is such that $[\vec{v}]_B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, then by definition

$$\vec{v} = 0 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2 + \dots + 0 \cdot \vec{v}_p = \vec{0}_V \quad \checkmark .$$

<u>T is onto</u>: Given any $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix}$ in $\mathbb{R}^p$, define

$$\vec{v} = a_1 \vec{v}_1 + a_2 \vec{v}_2 + \dots + a_p \vec{v}_p . \quad \text{Then } [\vec{v}]_B = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix}.$$

So every element of $\mathbb{R}^p$ comes from some element of $V$, i.e., $T$ is onto.

(25.2) <u>Invertible linear transformations</u> (also called <u>isomorphisms</u>*)

A <u>linear transformation</u> $T : V \to W$ between two vector spaces $V$ and $W$ which is both <u>one-to-one</u> and <u>onto</u> is called <u>invertible</u>. The reason is:

$T$ is one-to-one $\longleftrightarrow$ $\vec{v}_1 \neq \vec{v}_2$ implies $T(\vec{v}_1) \neq T(\vec{v}_2)$

$T$ is onto $\longleftrightarrow$ every $\vec{w}$ in $W$ comes from some $\vec{v}$ in $V$

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* From Greek: iso means same ; morph means shape
(ισο) (μορφ)

Combining the two properties, we can say:

$T$ is one-to-one and onto $\longleftrightarrow$ Every $\vec{w}$ in $W$ comes from <u>exactly one</u> $\vec{v}$ in $V$ (unique)

In other words, <u>for every $\vec{w}$ in $W$</u>, <u>there is a unique</u> $\vec{v}$ in $V$ such that $T(\vec{v}) = \vec{w}$.

So, we can define "return to the sender" map $\boxed{S : W \to V}$

$$S(\vec{w}) = \vec{v} \quad \text{if } \vec{v} \text{ is the unique vector in } V \text{ for which } T(\vec{v}) = \vec{w}.$$

It is clear that $S(T(\vec{v})) = \vec{v}$ for every $\vec{v}$ in $V$

$T(S(\vec{w})) = \vec{w}$ for every $\vec{w}$ in $W$.

We only <u>have to verify that $S$ is a linear transformation</u> (*-optional).

(i) If $\vec{w}_1, \vec{w}_2$ are in $W$, and $S(\vec{w}_1) = \vec{v}_1$, $S(\vec{w}_2) = \vec{v}_2$

(meaning $T(\vec{v}_1) = \vec{w}_1$ and $T(\vec{v}_2) = \vec{w}_2$)

then $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2) = \vec{w}_1 + \vec{w}_2$

Meaning $\vec{v}_1 + \vec{v}_2$ is the unique vector going to $\vec{w}_1 + \vec{w}_2$ via $T$.

By definition of $S$, we get $S(\vec{w}_1 + \vec{w}_2) = \vec{v}_1 + \vec{v}_2 = S(\vec{w}_1) + S(\vec{w}_2)$. ✓

(ii) $S(c\vec{w}) = c\,S(\vec{w})$ is checked similarly:

If $S(\vec{w}) = \vec{v}$ then $\vec{v}$ is the only vector for which $T(\vec{v}) = \vec{w}$. As $T$ is a linear transformation $T(c\vec{v}) = cT(\vec{v}) = c\vec{w}$. So $c\vec{v}$ is the unique vector going to $c\vec{w}$ via $T$. By definition of $S$, $S(c\vec{w}) = c\vec{v}$
$= c\,S(\vec{w})$.

(25.3) <u>Identity transformation</u>. - for a vector space $V$, we denote by $Id_V : V \longrightarrow V$ the linear transformation

$$Id_V(\vec{v}) = \vec{v} \text{ for every } \vec{v} \text{ in } V,$$

called the identity transformation.

The calculation of the last paragraph is summarized as:

<u>Theorem.</u> The following assertions are equivalent to each other, for a linear transformation $T : V \to W$.

(i) $T$ is both one-to-one and onto.

(ii) For every $\vec{w}$ in $W$, there is unique $\vec{v}$ in $V$ so that

$$T(\vec{v}) = \vec{w}$$

(iii) There is a linear transformation $S : W \to V$ such that

$$S(T(\vec{v})) = \vec{v}$$

$$T(S(\vec{w})) = \vec{w}$$

(for every $\vec{v}$ in $V$, $\vec{w}$ in $W$)

(written as $S \circ T = Id_V$

$T \circ S = Id_W$)

little circle stands for "composition" - see next page

The example from (25.1) above can be rephrased as:

"Coordinates relative to a basis" is an invertible linear transformation.

## (25.4) Composition of linear transformations

Assume we have three vector spaces, and linear transformations

$$T_1 : U \to V \quad ; \quad T_2 : V \to W.$$

We can "compose" these just as for ordinary functions.

$$T_2 \circ T_1 : U \to W$$

(read right to left - apply $T_1$ first and then $T_2$).

$$\boxed{T_2 \circ T_1 (\vec{u}) = T_2(T_1(\vec{u}))} \quad \text{for any } \vec{u} \text{ in } U.$$

## (25.5) Addition and scalar multiplication for linear transformations

Just as for matrices, we can add two linear transformations between same vector spaces, and scale a linear transformation.

$F, G : V \to W$

two linear transformations from $V$ to $W$

$\rightsquigarrow$

$F + G : V \to W$

$(F+G)(\vec{v}) = F(\vec{v}) + G(\vec{v})$

$\vec{v}$

for any $\vec{v}$ in $V$.

$F : V \to W$ linear transformation $\rightsquigarrow$ $cF : V \to W$

$c \in \mathbb{R}$ $\qquad (cF)(\vec{v}) = c \cdot F(\vec{v})$

## (25.6) Matrix representation of a linear transformation.

Assume $V$ and $W$ are two vector spaces.

Let $n = \dim(V)$ and $m = \dim(W)$.

Let $B = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} \subset V$ be a basis of $V$.

$C = \{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_m\} \subset W$ be a basis of $W$.

Given a linear transformation $T : V \to W$, we get an $m \times n$ matrix $A$ as follows

$$\underbrace{A}_{\substack{\text{matrix representation of} \\ T \text{ relative to the bases} \\ B \text{ of } V,\ C \text{ of } W.}} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ & & \vdots & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad \longleftrightarrow \quad \begin{aligned} T(\vec{v}_1) &= a_{11}\vec{w}_1 + a_{21}\vec{w}_2 + \dots + a_{m1}\vec{w}_m \\ T(\vec{v}_2) &= a_{12}\vec{w}_1 + a_{22}\vec{w}_2 + \dots + a_{m2}\vec{w}_m \\ &\vdots \\ T(\vec{v}_n) &= a_{1n}\vec{w}_1 + a_{2n}\vec{w}_2 + \dots + a_{mn}\vec{w}_m \end{aligned}$$

See Lecture 20, Sections (20.2) and (20.3) for a quick review and examples.