(26.0) Recall - last time we discussed linear transformations which are both <u>one-to-one</u> and <u>onto</u>. We verified that if $T: V \to W$ is such a linear transformation, then we can find $S: W \to V$ so that

$$S \circ T = Id_V \quad \text{and} \quad T \circ S = Id_W .$$

In this case, we say that T is <u>invertible</u>.

We also introduced the idea of <u>matrix representation of a linear transformation</u>:

Let V and W be two vector spaces; $n = \dim(V)$ and $m = \dim(W)$. Let $B = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} \subset V$ be a basis of V and $C = \{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_m\} \subset W$ a basis of W.

For a linear transformation $T: V \to W$, we found an $m \times n$ matrix A as follows:

$$\begin{aligned} T(\vec{v}_1) &= a_{11}\vec{w}_1 + a_{21}\vec{w}_2 + \dots + a_{m1}\vec{w}_m \\ T(\vec{v}_2) &= a_{12}\vec{w}_1 + a_{22}\vec{w}_2 + \dots + a_{m2}\vec{w}_m \\ &\vdots \\ T(\vec{v}_n) &= a_{1n}\vec{w}_1 + a_{2n}\vec{w}_2 + \dots + a_{mn}\vec{w}_m \end{aligned} \quad \rightsquigarrow \quad A = \begin{bmatrix} a_{11} & a_{12} & & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}$$

$m \times n$ matrix

We say A represents T in the bases B (of V) and C (of W).

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

coordinates of $T(\vec{v}_1)$ relative to the basis C $\quad [T(\vec{v}_1)]_C$

$[T(\vec{v}_2)]_C$

$[T(\vec{v}_n)]_C$

Thus, A is computed by evaluating T on the vectors from the basis B of V, and working out their coordinates relative to the basis C of W.

We will denote this matrix by $[T]_{C,B}$.

$$\boxed{\text{Linear transformations } V \longrightarrow W} \longrightarrow \boxed{m \times n \text{ matrices}}$$

$\begin{pmatrix} m = \dim(W) \\ n = \dim(V) \end{pmatrix}$

depends on a choice of a basis of V and a basis of W

(26.1) Example. Let $V = M_{2\times 2}(\mathbb{R})$ (vector space of all 2×2 matrices).

$W = P_2$ (vector space of polynomials of degree at most 2).

$F : V \longrightarrow W$ given by $F\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = d + (b-c)x + ax^2$

Basis of V: $B = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ (recall: E_{ij} is the matrix which has 1 at (i,j)-spot and zeroes everywhere else).

Basis of W: $C = \{1, x, x^2\}$.

Compute $[F]_{C,B}$ (a 3×4 matrix).

Sol. We have to compute $F(E_{11})$, $F(E_{12})$, $F(E_{21})$ and $F(E_{22})$.

$F\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = 0 + 0\cdot x + x^2 \rightsquigarrow 1^{st}$ column of $[F]_{C,B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$F\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = 0 + 1\cdot x + 0\cdot x^2 \rightsquigarrow 2^{nd}$ " " " $= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$F\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = 0 - 1\cdot x + 0\cdot x^2 \rightsquigarrow 3^{rd}$ " " " $= \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$

$F\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = 1 + 0\cdot x + 0\cdot x^2 \rightsquigarrow 4^{th}$ " " " $= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Answer: $[F]_{C,B} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.

3×4.

dim W dim V

Continuing the example from the last page; let us see how the computation of $F(X)$ (X is a 2×2 matrix, $X \in V$) relates to $[F]_{G,B}$ (matrix mult.)

Let $X = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$. We can directly see that (from the formula of F)

$$F(X) = 4 + (1-(-1))x + 2x^2 = 4 + 2x + 2x^2.$$

On the other hand $[X]_B = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 4 \end{bmatrix} \in \mathbb{R}^4.$

(coordinates of X relative to the basis B)

- $$[F]_{G,B} \cdot [X]_B = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$$

- $$[F(X)]_G = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$$ $= \checkmark$

coordinates of $F(X) = 4 + 2x + 2x^2$ in the basis G

It is true in general: $\boxed{[F]_{G,B} \cdot [\vec{v}]_B = [F(\vec{v})]_G}$.

(26.2) Properties of matrix representation.

I. Let $T : V \to W$ be a linear transformation from a vector space V to another vector space W.

Let $B_V \subset V$ and $B_W \subset W$ be two bases. Then for every $\vec{v}$ in V :

$$\boxed{\underset{m \times 1}{\underbrace{[T(\vec{v})]_{B_W}}} = \underset{m \times n}{\underbrace{[T]_{B_W, B_V}}} \cdot \underset{n \times 1}{\underbrace{[\vec{v}]_{B_V}}}} \quad \text{matrices}$$

$(\dim(W) = m$ and $\dim(V) = n)$.

In pictures

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "{[\ ]_{B_V}}"'] & W \arrow[d, "{[\ ]_{B_W}}"] \\ \mathbb{R}^n \arrow[r, "{[T]_{B_W, B_V}}"'] & \mathbb{R}^m \end{tikzcd}$$

(Proof (Optional). - If $B_V = \{\vec{v}_1, \dots, \vec{v}_n\}$ and $B_W = \{\vec{w}_1, \dots, \vec{w}_m\}$, then

$$[T]_{B_W, B_V} = \Big[\underset{1^{st}\text{ col.}}{[T(\vec{v}_1)]_{B_W}} \,\Big\vdots\, \underset{2^{nd}\text{ col.}}{[T(\vec{v}_2)]_{B_W}} \,\Big\vdots\, \cdots \,\Big\vdots\, \underset{n^{th}\text{ col.}}{[T(v_n)]_{B_W}} \Big].$$

Let $\vec{v} = a_1 \vec{v}_1 + a_2 \vec{v}_2 + \dots + a_n \vec{v}_n$ be a vector in V.

Then, using the fact that $[\ ]_{B_W}$ is a linear transformation,

$$[T(\vec{v})]_{B_W} = a_1 [T(\vec{v}_1)]_{B_W} + a_2 [T(\vec{v}_2)]_{B_W} + \dots + a_n [T(\vec{v}_n)]_{B_W}$$

and $[T]_{B_W, B_V} \cdot [\vec{v}]_{B_V} = \left[[T(\vec{v}_1)]_{B_W} \ \cdots \ [T(\vec{v}_n)]_{B_W} \right] \cdot \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$

$$= a_1 [T(\vec{v}_1)]_{B_W} + a_2 [T(\vec{v}_2)]_{B_W} + \cdots + a_n [T(\vec{v}_n)]_{B_W}.$$

So, the two sides of the equation $[T(\vec{v})]_{B_W} = [T]_{B_W B_V} [\vec{v}]_{B_V}$ are equal ✓.)

II. Addition and scalar multiplication:

$$[T_1 + T_2]_{B_W B_V} = [T_1]_{B_W B_V} + [T_2]_{B_W B_V}$$

$$[cT]_{B_W B_V} = c\,[T]_{B_W B_V}.$$

(here: T_1, T_2, T are linear transformations $V \to W$; $c \in \mathbb{R}$).

III. Composition = Matrix Multiplication

If V, W and X are three vector spaces,

$T_1 : V \to W$ and $T_2 : W \to X$ are linear trans.

B_V, B_W and B_X are bases of V, W and X respectively.

Then

$$\boxed{\underbrace{[T_2 \circ T_1]_{B_X B_V}}_{\text{Composition } T_2 \circ T_1 : V \to X} = [T_2]_{B_X B_W} \underbrace{\cdot}_{\text{matrix multiplication}} [T_1]_{B_W B_V}}$$

IV. $T: V \to W$ linear transformation

$B_V \subset V$ a basis of V ; $B_W \subset W$ a basis of W.

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "{[\ ]_{B_V}}"'] & W \arrow[d, "{[\ ]_{B_W}}"] \\ \mathbb{R}^n \arrow[r, "{A = [T]_{B_W B_V}}"'] & \mathbb{R}^m \end{tikzcd}$$

(m×n matrix)

$m = \dim(W)$.
$n = \dim(V)$.

- $\vec{v} \in N(T)$ if and only if $[\vec{v}]_{B_V} \in N(A)$
  (null space of T) (null space of A)
- $\vec{w} \in R(T)$ if and only if $[\vec{w}]_{B_W} \in R(A)$
  (range of T) (range of A).

$$\boxed{\text{Nullity}(T) = \text{Nullity}(A)} \qquad \boxed{\text{Rank}(T) = \text{Rank}(A)}$$

Rank-Nullity Theorem :

for matrices : $\boxed{\text{Rank}(A) + \text{Nullity}(A) = \#\,\text{Cols.}(A)}$

for linear transformation $T: V \to W$: $\boxed{\text{Rank}(T) + \text{Nullity}(T) = \dim(V)}$