

(26.0) Recall - last time we discussed linear transformations which are both one-to-one and onto. We verified that

if  $T: V \rightarrow W$  is such a linear transformation, then we can find  $S: W \rightarrow V$  so that

$$S \circ T = \text{Id}_V \quad \text{and} \quad T \circ S = \text{Id}_W.$$

In this case, we say that  $T$  is invertible.

We also introduced the idea of matrix representation of a linear transformation:

Let  $V$  and  $W$  be two vector spaces;  $n = \dim(V)$  and  $m = \dim(W)$ . Let  $B = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} \subset V$  be a basis of  $V$  and  $C = \{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_m\} \subset W$  a basis of  $W$ .

For a linear transformation  $T: V \rightarrow W$ , we found an  $m \times n$  matrix  $A$  as follows:

$$T(\vec{v}_1) = a_{11}\vec{w}_1 + a_{21}\vec{w}_2 + \dots + a_{m1}\vec{w}_m$$

$$T(\vec{v}_2) = a_{12}\vec{w}_1 + a_{22}\vec{w}_2 + \dots + a_{m2}\vec{w}_m$$

$$\vdots$$

$$T(\vec{v}_n) = a_{1n}\vec{w}_1 + a_{2n}\vec{w}_2 + \dots + a_{mn}\vec{w}_m$$

$$\leadsto A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

$m \times n$  matrix

We say A represents T in the bases B (of V) and C (of W).

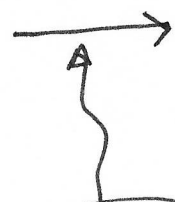
$$A = \begin{bmatrix} \begin{matrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{matrix} & \begin{matrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{matrix} & \dots & \begin{matrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{matrix} \end{bmatrix}$$

$\begin{matrix} \uparrow & \uparrow & & \uparrow \\ \text{coordinates of } T(\vec{v}_1) \text{ relative to the basis } C & [T(\vec{v}_2)]_C & & [T(\vec{v}_n)]_C \\ [T(\vec{v}_1)]_C & & & \end{matrix}$

Thus, A is computed by evaluating T on the vectors from the basis B of V, and working out their coordinates relative to the basis C of W.

We will denote this matrix by  $[T]_{C,B}$ .

Linear transformations  
 $V \rightarrow W$



$m \times n$  matrices

$(m = \dim(W))$   
 $(n = \dim(V))$

depends on a choice of a basis of V and a basis of W

(26.1) Example. Let  $V = M_{2 \times 2}(\mathbb{R})$  (vector space of all  $2 \times 2$  matrices). (3)

$W = P_2$  (vector space of polynomials of degree at most 2).

$F: V \longrightarrow W$  given by  $F\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = d + (b-c)x + ax^2$

Basis of  $V$ :  $B = \{E_{11}, E_{12}, E_{21}, E_{22}\}$  (recall:  $E_{ij}$  is the matrix which has 1 at  $(i,j)$ -spot and zeroes everywhere else).

Basis of  $W$ :  $C = \{1, x, x^2\}$ .

Compute  $[F]_{C,B}$  (a  $3 \times 4$  matrix).

Sol. We have to compute  $F(E_{11}), F(E_{12}), F(E_{21})$  and  $F(E_{22})$ .

$$F\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = 0 + 0 \cdot x + x^2 \quad \leadsto \quad 1^{\text{st}} \text{ column of } [F]_{C,B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$F\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = 0 + 1 \cdot x + 0 \cdot x^2 \quad \leadsto \quad 2^{\text{nd}} \text{ " " " " } = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$F\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = 0 - 1 \cdot x + 0 \cdot x^2 \quad \leadsto \quad 3^{\text{rd}} \text{ " " " " } = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

$$F\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = 1 + 0 \cdot x + 0 \cdot x^2 \quad \leadsto \quad 4^{\text{th}} \text{ " " " " } = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Answer:  $[F]_{C,B} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

$\begin{matrix} \uparrow & \uparrow \\ \dim W & \dim V \end{matrix}$

Continuing the example from the last page; let us see how the computation of  $F(X)$  ( $X$  is a  $2 \times 2$  matrix,  $X \in V$ ) relates to  $[F]_{G,B}$  (matrix mult.)

Let  $X = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$ . We can directly see that (from the formula of  $F$ )

$$F(X) = 4 + (1 - (-1))x + 2x^2 = 4 + 2x + 2x^2.$$

On the other hand  $[X]_B = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 4 \end{bmatrix} \in \mathbb{R}^4$ .  
 (coordinates of  $X$  relative to the basis  $B$ )

$$\bullet [F]_{G,B} \cdot [X]_B = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$$

$$\bullet [F(X)]_G = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$$

coordinates of  $F(X)$   
 $= 4 + 2x + 2x^2$   
 in the basis  $G$

It is true in general:  $[F]_{G,B} \cdot [\vec{v}]_B = [F(\vec{v})]_G$

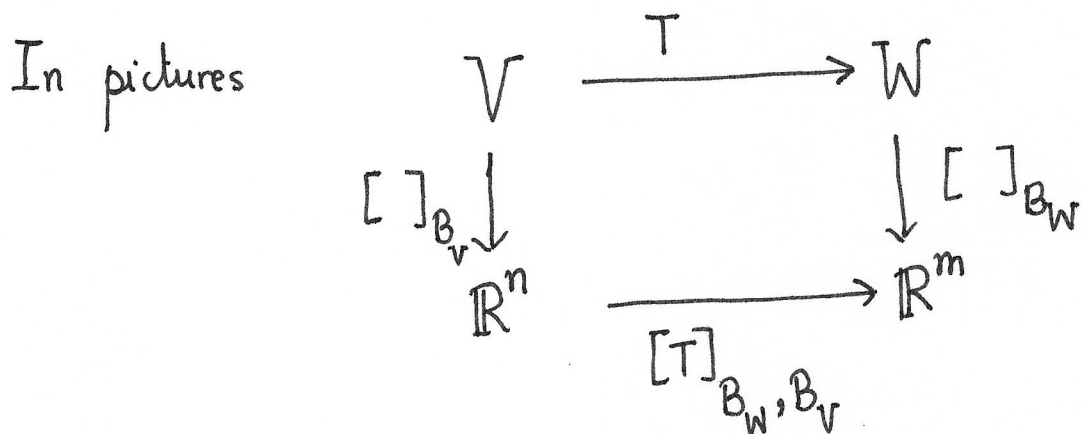
(26.2) Properties of matrix representation.

I. Let  $T: V \rightarrow W$  be a linear transformation from a vector space  $V$  to another vector space  $W$ .

Let  $B_V \subset V$  and  $B_W \subset W$  be two bases. Then for every

$$\vec{v} \text{ in } V: \quad \boxed{\begin{matrix} [T(\vec{v})]_{B_W} & = & [T]_{B_W, B_V} & \cdot & [\vec{v}]_{B_V} \\ \uparrow & & \uparrow & & \uparrow \\ m \times 1 & & m \times n & & n \times 1 \end{matrix}} \quad \text{matrices}$$

( $\dim(W) = m$  and  $\dim(V) = n$ ).



(Proof (Optional)). - If  $B_V = \{\vec{v}_1, \dots, \vec{v}_n\}$  and  $B_W = \{\vec{w}_1, \dots, \vec{w}_m\}$ , then

$$[T]_{B_W, B_V} = \begin{bmatrix} [T(\vec{v}_1)]_{B_W} & \vdots & [T(\vec{v}_2)]_{B_W} & \vdots & \dots & \vdots & [T(\vec{v}_n)]_{B_W} \\ \text{1st col.} & & \text{2nd col.} & & & & \text{nth col.} \end{bmatrix}$$

Let  $\vec{v} = a_1 \vec{v}_1 + a_2 \vec{v}_2 + \dots + a_n \vec{v}_n$  be a vector in  $V$ .

Then, using the fact that  $[\ ]_{B_W}$  is a linear transformation,

$$[T(\vec{v})]_{B_W} = a_1 [T(\vec{v}_1)]_{B_W} + a_2 [T(\vec{v}_2)]_{B_W} + \dots + a_n [T(\vec{v}_n)]_{B_W}$$

$$\text{and } [T]_{B_W, B_V} \cdot [\vec{v}]_{B_V} = \left[ [T(\vec{v}_1)]_{B_W} \cdots [T(\vec{v}_n)]_{B_W} \right] \cdot \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \quad (6)$$

$$= a_1 [T(\vec{v}_1)]_{B_W} + a_2 [T(\vec{v}_2)]_{B_W} + \cdots + a_n [T(\vec{v}_n)]_{B_W}$$

So, the two sides of the equation  $[T(\vec{v})]_{B_W} = [T]_{B_W, B_V} [\vec{v}]_{B_V}$  are equal  $\checkmark$ .)

II. Addition and scalar multiplication:

$$[T_1 + T_2]_{B_W, B_V} = [T_1]_{B_W, B_V} + [T_2]_{B_W, B_V}$$

$$[cT]_{B_W, B_V} = c [T]_{B_W, B_V}$$

(here:  $T_1, T_2, T$  are linear transformations  $V \rightarrow W$ ;  $c \in \mathbb{R}$ ).

III. Composition = Matrix Multiplication

If  $V, W$  and  $X$  are three vector spaces,

$T_1: V \rightarrow W$  and  $T_2: W \rightarrow X$  are linear trans.

$B_V, B_W$  and  $B_X$  are bases of  $V, W$  and  $X$  respectively.

Then

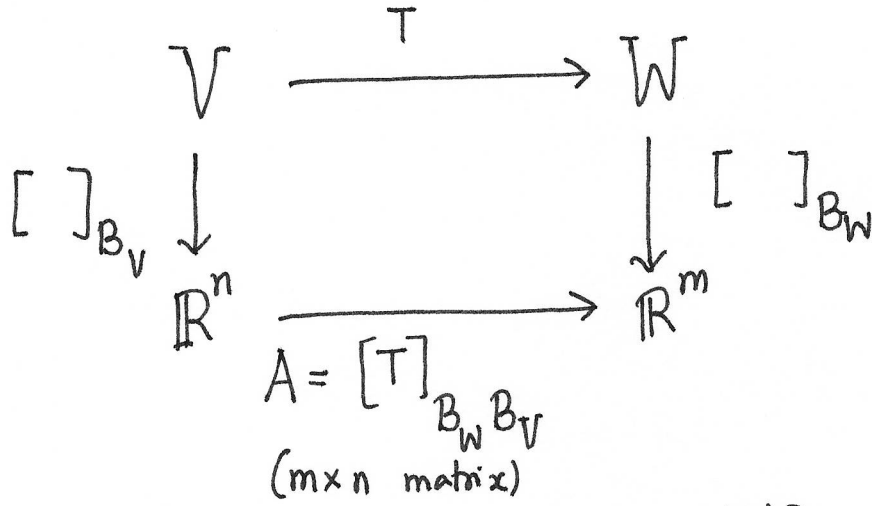
$$\boxed{[T_2 \circ T_1]_{B_X, B_V} = [T_2]_{B_X, B_W} \cdot [T_1]_{B_W, B_V}}$$

Composition  $T_2 \circ T_1: V \rightarrow X$

matrix multiplication

IV.  $T: V \rightarrow W$  linear transformation

$B_V \subset V$  a basis of  $V$  ;  $B_W \subset W$  a basis of  $W$ .



$m = \dim(W)$ .

$n = \dim(V)$ .

- $\vec{v} \in N(T)$  if and only if  $[\vec{v}]_{B_V} \in N(A)$   
 (null space of  $T$ ) (null space of  $A$ )
- $\vec{w} \in R(T)$  if and only if  $[\vec{w}]_{B_W} \in R(A)$   
 (range of  $T$ ) (range of  $A$ ).

$Nullity(T) = Nullity(A)$

$Rank(T) = Rank(A)$

Rank-Nullity Theorem :

for matrices :  $Rank(A) + Nullity(A) = \# \text{ Cols.}(A)$

for linear transformation :  $Rank(T) + Nullity(T) = \dim(V)$   
 $T: V \rightarrow W$