Lecture 27

Determinants

(27.0) In today's lecture we are going to define the determinant of a square matrix. That is, for any positive integer $n \geq 2$, we will define a function:

$$\mathrm{Det} : \underset{(n \times n \text{ matrices})}{M_{n\times n}(\mathbb{R})} \longrightarrow \mathbb{R}$$

This function is going to have the following properties:

(i) It is a polynomial in the entries of the matrix.

(ii) $\boxed{A \in M_{n\times n}(\mathbb{R}) \text{ is invertible}} \longleftrightarrow \boxed{\mathrm{Det}(A) \neq 0}$

(Determinant tests invertibility).

(iii) For any two (square) $n \times n$ matrices A and B

$$\boxed{\mathrm{Det}(AB) = \mathrm{Det}(A) \cdot \mathrm{Det}(B)}$$

(iv) $\boxed{\mathrm{Det}(A^T) = \mathrm{Det}(A)}$

(V) If $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ 0's & & \ddots & \vdots \\ & & & a_{nn} \end{bmatrix}$ (upper triangular matrix)

$\boxed{\text{all entries below the diagonal are } 0}$

then $\mathrm{Det}(A) = a_{11} a_{22} \cdots a_{nn}$ (product of the diagonal entries).

e.g. $\mathrm{Det}(I_n) = 1$ where I_n is the $n \times n$ identity matrix:

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & & \vdots & \\ 0 & 0 & & \cdots & 0 & 1 \end{bmatrix}$$

(27.1) $\boxed{n = 2}$ $\quad \mathrm{Det}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$.

(polynomial in a, b, c, d). Property (i)

(Property (ii)) • If $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is NOT invertible, then the two columns $\begin{bmatrix} a \\ c \end{bmatrix}$ and $\begin{bmatrix} b \\ d \end{bmatrix}$ are going to be linearly dependent.

So, either one of these vectors is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ so $ad - bc = 0$.

or $\begin{bmatrix} b \\ d \end{bmatrix} = k \cdot \begin{bmatrix} a \\ c \end{bmatrix}$ for some k $\leadsto$ $ad - bc = a(kc) - (ka)c = 0$.

- $\text{Det}\left(\begin{bmatrix} a & c \\ b & d \end{bmatrix}\right) = ad - cb = ad - bc$

$$= \text{Det}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right). \qquad \text{(Property (iv))}$$

- $\text{Det}\left(\begin{bmatrix} a & c \\ 0 & d \end{bmatrix}\right) = ad =$ product of diagonal entries (Property (v)).

Let us check Property (iii) directly*.

$$A_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$$

$$A_1 A_2 = \begin{bmatrix} a_1 a_2 + b_1 c_2 & a_1 b_2 + b_1 d_2 \\ c_1 a_2 + d_1 c_2 & c_1 b_2 + d_1 d_2 \end{bmatrix}$$

$$\begin{aligned}
\text{Det}(A_1 A_2) &= (a_1 a_2 + b_1 c_2)(c_1 b_2 + d_1 d_2) - (a_1 b_2 + b_1 d_2)(c_1 a_2 + d_1 c_2) \\
&= \cancel{a_1 a_2 c_1 b_2} + \underline{a_1} a_2 \underline{d_1} d_2 + b_1 c_2 c_1 b_2 + \cancel{b_1 c_2 d_1 d_2} \\
&\quad - \cancel{a_1 b_2 c_1 a_2} - \underline{a_1} b_2 \underline{d_1} c_2 - b_1 d_2 c_1 a_2 - \cancel{b_1 d_2 d_1 c_2} \\
&= a_1 d_1 (a_2 d_2 - b_2 c_2) - b_1 c_1 (a_2 d_2 - b_2 c_2) \\
&= (a_1 d_1 - b_1 c_1)(a_2 d_2 - b_2 c_2) = \text{Det}(A_1) \cdot \text{Det}(A_2).
\end{aligned}$$

$$\boxed{\text{Det}(A_1 A_2) = \text{Det}(A_1)\,\text{Det}(A_2)}$$

---

*Optional

(27.2) Recursive definition of determinant

Let A be an $n \times n$ matrix.

For each (i, j) we define an $(n-1) \times (n-1)$ matrix

row index   column index M_{ij} by deleting the i^{th} row of A and j^{th} column of A.

e.g. $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 4 \\ -1 & 3 & 7 \end{bmatrix} \rightsquigarrow \begin{cases} M_{11} = \begin{bmatrix} 0 & 4 \\ 3 & 7 \end{bmatrix} \\ M_{21} = \begin{bmatrix} 2 & 1 \\ 3 & 7 \end{bmatrix} \\ M_{12} = \begin{bmatrix} 2 & 4 \\ -1 & 7 \end{bmatrix} \end{cases}$ and so on.

Let $A_{ij} = (-1)^{i+j} \, \text{Det}(M_{ij})$. These numbers are called cofactors.

Definition. $\boxed{\text{Det}(A) = a_{11} \cdot A_{11} + a_{12} A_{12} + \cdots + a_{1n} A_{1n}}$

Remark.- This definition is recursive in nature. Meaning, in order to define determinant of an $n \times n$ matrix, it is assumed that we know the definition of the determinant of an $(n-1)\times(n-1)$ matrix.

Example. $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 4 \\ -1 & 3 & 7 \end{bmatrix}$ 3x3 matrix from the last page

$$A_{11} = (-1)^{1+1} \cdot \text{Det}(M_{11}) = +\text{Det}\left(\begin{bmatrix} 0 & 4 \\ 3 & 7 \end{bmatrix}\right) = 0(7)-4(3) = -12.$$

$$A_{12} = (-1)^{1+2}\,\text{Det}(M_{12}) = -\text{Det}\left(\begin{bmatrix} 2 & 4 \\ -1 & 7 \end{bmatrix}\right) = -\left(2(7)-4(-1)\right) = -18$$

$$A_{13} = (-1)^{1+3}\,\text{Det}(M_{13}) = +\text{Det}\left(\begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}\right) = 6.$$

So,
$$\begin{aligned}\text{Det}(A) &= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} \\ &= 1(-12) + 2(-18) + 1(6) = -12-36+6 \\ &= -42.\end{aligned}$$

(27.3) How to ~~remb~~ remember this definition?

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ be an $n\times n$ matrix.

$$\text{Det}(A) = (+1)\cdot a_{11}\cdot \text{Det}\begin{bmatrix} a_{11} - a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \vdots & \vdots & \\ a_{n1} & a_{n2} \cdots & a_{nn}\end{bmatrix}$$

Matrix obtained by deleting 1^{st} row and 1^{st} column

$$+ (-1)\, a_{12}\, \text{Det}\begin{bmatrix} a_{11} - a_{12} & \cdots & a_{1n} \\ a_{21} \quad a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \\ a_{n1} \quad a_{n2} & \cdots & a_{nn}\end{bmatrix}$$

Matrix obtained by deleting 1^{st} row and 2^{nd} column

signs alternate

$$+ \cdots + (-1)^{1+n}\, a_{1n}\, \text{Det}\begin{bmatrix} a_{11} - a_{12} & \cdots & a_{1n} \\ a_{21} \quad a_{22} & \cdots & a_{2n} \\ \vdots & & \\ a_{n1} \quad a_{n2} & \cdots & a_{nn}\end{bmatrix}$$

Matrix obtained by deleting 1^{st} row & n^{th} col.

(27.4) Example. – Compute the determinant of

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 \\ -1 & 2 & 3 & 1 \\ -3 & 2 & -1 & 0 \\ 2 & -3 & -2 & 1 \end{bmatrix}$$

$$\text{Det}(A) = 1\cdot \text{Det}\begin{bmatrix} 2 & 3 & 1 \\ 2 & -1 & 0 \\ -3 & -2 & 1\end{bmatrix} - 2\,\text{Det}\begin{bmatrix} -1 & 3 & 1 \\ -3 & -1 & 0 \\ 2 & -2 & 1\end{bmatrix}$$

$$\underbrace{+\,0\cdot \text{Det}\begin{bmatrix} -1 & 2 & 1 \\ -3 & 2 & 0 \\ 2 & -3 & 1\end{bmatrix}}_{\text{don't need to compute this one.}} - 2\,\text{Det}\begin{bmatrix} -1 & 2 & 3 \\ -3 & 2 & -1 \\ 2 & -3 & -2\end{bmatrix}$$

$$\mathrm{Det}\begin{bmatrix} 2 & 3 & 1 \\ 2 & -1 & 0 \\ -3 & -2 & 1 \end{bmatrix} = 2 \cdot \mathrm{Det}\begin{bmatrix} -1 & 0 \\ -2 & 1 \end{bmatrix} - 3\,\mathrm{Det}\begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix} + 1\,\mathrm{Det}\begin{bmatrix} 2 & -1 \\ -3 & -2 \end{bmatrix}$$

$$= 2(-1) - 3(2) + 1(-7) = -2 - 6 - 7 = -15$$

$$\mathrm{Det}\begin{bmatrix} -1 & 3 & 1 \\ -3 & -1 & 0 \\ 2 & -2 & 1 \end{bmatrix} = (-1)\,\mathrm{Det}\begin{bmatrix} -1 & 0 \\ -2 & 1 \end{bmatrix} - 3\,\mathrm{Det}\begin{bmatrix} -3 & 0 \\ 2 & 1 \end{bmatrix} + 1 \cdot \mathrm{Det}\begin{bmatrix} -3 & -1 \\ 2 & -2 \end{bmatrix}$$

$$= (-1)(-1) - 3(-3) + 1(8) = 1 + 9 + 8 = 18.$$

$$\mathrm{Det}\begin{bmatrix} -1 & 2 & 3 \\ -3 & 2 & -1 \\ 2 & -3 & -2 \end{bmatrix} = (-1)\,\mathrm{Det}\begin{bmatrix} 2 & -1 \\ -3 & -2 \end{bmatrix} - 2\,\mathrm{Det}\begin{bmatrix} -3 & -1 \\ 2 & -2 \end{bmatrix} + 3\,\mathrm{Det}\begin{bmatrix} -3 & 2 \\ 2 & -3 \end{bmatrix}$$

$$= (-1)(-7) - 2(+8) + 3(5)$$

$$= 7 - 16 + 15 = 6.$$

So $\mathrm{Det}(A) = 1 \cdot (-15) - 2(18) - 2(6)$

$$= -15 - 36 - 12 = -63.$$

(27.5) Since our definition of the determinant is recursive, any claim about determinants is proved using <u>an induction argument</u> - (meaning:
- check it for 2×2 case.
- assume it for $(n-1) \times (n-1)$ case
- check it for $(n \times n)$ case)

For instance: If the 1st column of A is $\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, then

$$\operatorname{Det}(A) = 0.$$

Proof. $\operatorname{Det} \begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix} = 0$ ✓ So the claim is true for 2×2 matrices.

Let us assume it is true for all $(n-1)\times(n-1)$ matrices.

If $A = \begin{bmatrix} 0 & a_{12} & & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & a_{n2} & & a_{nn} \end{bmatrix}$ is $n\times n$ size matrix with 1st column $= \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$,

then
$$\operatorname{Det}(A) = 0 \cdot \operatorname{Det} \begin{bmatrix} a_{22} & \cdots & a_{2n} \\ & & \vdots \\ a_{n2} & \cdots & a_{nn} \end{bmatrix} - a_{12} \operatorname{Det} \begin{bmatrix} 0 & a_{23} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ 0 & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

zero

$$+ a_{13} \cdot \operatorname{Det} \begin{bmatrix} 0 & a_{22} & a_{24} & \cdots & a_{2n} \\ \vdots & & & & \vdots \\ 0 & a_{n2} & a_{n4} & \cdots & a_{nn} \end{bmatrix} \quad \cdots$$

all terms here involve determinants of $(n-1)\times(n-1)$ matrices whose 1st col. $= \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$$= 0.$$

(27.6) (Property (V) - page 2)

$$\mathrm{Det}\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ & & \vdots & & \\ 0 & - & - & 0 & a_{nn} \end{bmatrix}\right) = a_{11}a_{22}\cdots a_{nn}$$

all 0's below the diagonal

(product of diagonal entries).

Reason:

$$\mathrm{Det}\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ & & & & \vdots \\ 0 & \cdots & \cdots & 0 & a_{nn} \end{bmatrix} = a_{11}\cdot \mathrm{Det}\begin{bmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ & a_{33} & \cdots & a_{3n} \\ \text{0's} & & \ddots & \vdots \\ \text{below the diagonal} & & & a_{nn} \end{bmatrix}$$

+ all other terms involve determinant of matrices whose 1^{st} col. $= \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, so zero by the argument on the last page.

Continuing this way, we get:

$$\mathrm{Det}\begin{bmatrix} a_{11} & & * \\ & \ddots & \\ \text{0's} & & a_{nn} \end{bmatrix} = a_{11}\cdot \mathrm{Det}\begin{bmatrix} a_{22} & & * \\ & \ddots & \\ \text{0's} & & a_{nn} \end{bmatrix} = a_{11}a_{22}\cdot \mathrm{Det}\begin{bmatrix} a_{33} & & * \\ & \ddots & \\ \text{0's} & & a_{nn} \end{bmatrix}$$

$$= \cdots = \boxed{a_{11}a_{22}a_{33}\cdots a_{nn}}.$$