Lecture 28

(28.0) Recall that last time we defined <u>the determinant</u>.

A : $n\times n$ (square) matrix $\rightsquigarrow$ $\mathrm{Det}(A)$ is a number.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ & & \vdots & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \in M_{n\times n}(\mathbb{R}),$$

$\mathrm{Det}(A)$ is defined in terms of determinant of $(n-1)\times(n-1)$ submatrices of A.

$$\mathrm{Det}(A) = a_{11}\cdot \mathrm{Det}\begin{bmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ a_{32} & a_{33} & \cdots & a_{3n} \\ & \vdots & & \\ a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \quad (1^{st} \text{ row } \& \ 1^{st} \text{ col. of } A \text{ deleted})$$

$$- a_{12}\cdot \mathrm{Det}\begin{bmatrix} a_{21} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{33} & \cdots & a_{3n} \\ & \vdots & & \\ a_{n1} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \quad (1^{st} \text{ row } \& \ 2^{nd} \text{ col. deleted})$$

$$+ a_{13}\cdot \mathrm{Det}\begin{bmatrix} a_{21} & a_{22} & a_{24} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{34} & \cdots & a_{3n} \\ & & \vdots & & \\ a_{n1} & a_{n2} & a_{n4} & \cdots & a_{nn} \end{bmatrix} \quad (1^{st} \text{ row } \& \ 3^{rd} \text{ col. deleted})$$

$$\cdots + (-1)^{1+n} a_{1n} \,\mathrm{Det}\begin{bmatrix} a_{22} & a_{23} & \cdots & a_{2,n-1} \\ a_{32} & a_{33} & \cdots & a_{3,n-1} \\ & \vdots & & \\ a_{n2} & a_{n3} & \cdots & a_{n,n-1} \end{bmatrix} \quad (1^{st} \text{ row } \& \ n^{th} \text{ col. deleted})$$

(so on)

Determinant of a 2×2 matrix:

$$\text{Det} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

Example $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ $(\text{Det}(A) = 0)$

$$\text{Det}(A) = 1 \cdot \text{Det} \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix} - 2 \,\text{Det} \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix} + 3 \,\text{Det} \begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}$$

$$= 1 \cdot (45 - 48) - 2\,(36 - 42) + 3\,(32 - 35)$$

$$= 1(-3) - 2(-6) + 3(-3) = -3 + 12 - 9 = 0.$$

Hence A is not invertible.

(Check: $\text{Col}\,1 + \text{Col}\,3 = 2 \cdot (\text{Col}\,2)$ - so columns are not linearly independent).

Today we are going to discuss how the determinant changes <u>under row operations.</u>

Recall - Last time we also saw that the determinant of a <u>triangular</u> matrix is equal to the product of diagonal entries.

(all entries below the diagonal are 0 $\longleftrightarrow$ upper triangular
" " above " " " " $\longleftrightarrow$ lower triangular)

So, the knowledge of the behaviour of determinant under row operations will <u>significantly</u> simplify our calculations.

(28.1) <u>Elementary row operations</u>

Recall that there are 3 elementary row operations.

I. Swap. - for $i \neq j$, we can flip i^{th} row $\longleftrightarrow$ j^{th} row.

II. Scale (by $\alpha \neq 0$) - we can change i^{th} row to α times i^{th} row.

III. Combine. - for $i \neq j$, and a scalar λ, we can replace i^{th} row by $(i^{th}\text{ row}) + \lambda\,(j^{th}\text{ row})$.

Assume A is an $n \times n$ matrix and B is obtained from A by an elementary row operation.

Operation	Determinant relation
Swap	$\mathrm{Det}(B) = -\mathrm{Det}(A)$
Scale by $\alpha \neq 0$	$\mathrm{Det}(B) = \alpha \cdot \mathrm{Det}(A)$
Combine	$\mathrm{Det}(B) = \mathrm{Det}(A)$

[Table of how determinant changes under row operations.]

Example. $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 4 \\ -1 & 3 & 7 \end{bmatrix}$. Compute $\mathrm{Det}(A)$ by putting A into its ~~reduced~~ echelon form

$$\underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 4 \\ -1 & 3 & 7 \end{bmatrix}}_{A_1} \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 2R_1} \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & 2 \\ 0 & 5 & 8 \end{bmatrix}}_{A_2} \xrightarrow[R_2 \to \left(\frac{-1}{4}\right) \cdot R_2]{} \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & \frac{-1}{2} \\ 0 & 5 & 8 \end{bmatrix}}_{A_3}$$

$$\downarrow R_3 \to R_3 - 5R_2$$

$$\underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{bmatrix}}_{A_5} \xleftarrow[R_3 \to \left(\frac{2}{21}\right) R_3]{} \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1/2 \\ 0 & 0 & \underbrace{8 + \frac{5}{2}}_{= \frac{21}{2}} \end{bmatrix}}_{A_4}$$

- $\mathrm{Det}(A_2) = \det(A_1) = \mathrm{Det}(A)$
- $\mathrm{Det}(A_3) = \frac{-1}{4} \cdot \mathrm{Det}(A_2) = \frac{-1}{4} \mathrm{Det}(A)$
- $\mathrm{Det}(A_4) = \mathrm{Det}(A_3) = \frac{-1}{4} \mathrm{Det}(A)$
- $\mathrm{Det}(A_5) = \frac{2}{21} \mathrm{Det}(A_4) = \frac{2}{21} \cdot \left(\frac{-1}{4}\right) \mathrm{Det}(A)$.

Now $\mathrm{Det}(A_5) = 1$.

So $\mathrm{Det}(A) = \frac{21}{2} \cdot (-4) = -42$.

(28.2) Summarizing: given an $n \times n$ matrix A,

- Use Gauss-Jordan algorithm to put A into echelon form

$$A \dashrightarrow B \text{ (echelon form) (or reduced echelon)}$$

- k swaps
- l scales by non-zero scalars $c_1, c_2, \ldots, c_l$

$$\boxed{\mathrm{Det}(A) = \frac{(-1)^k}{c_1 c_2 \cdots c_l} \mathrm{Det}(B)}$$

This immediately proves that $\boxed{A \text{ is } \underline{\text{invertible}} \text{ if and only if } \mathrm{Det}(A) \neq 0.}$

(reason: A invertible means its echelon form is

$$B = \begin{bmatrix} 1 & & & * \\ & 1 & & \\ & & 1 & \\ 0's & & \ddots & 1 \end{bmatrix} \rightsquigarrow \mathrm{Det}(B) = 1.)$$

(28.3) Another example.

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 4 & 8 & 4 \\ -3 & 2 & -1 & 0 \\ 2 & -3 & -2 & 1 \end{bmatrix}$$

Compute $\mathrm{Det}(A)$.

$$\begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 4 & 8 & 4 \\ -3 & 2 & -1 & 0 \\ 2 & -3 & -2 & 1 \end{bmatrix} \xrightarrow[R_4 \to R_4 - 2R_1]{R_3 \to R_3 + 3R_1} \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 4 & 8 & 4 \\ 0 & 8 & -1 & 6 \\ 0 & -7 & -2 & -3 \end{bmatrix} \xrightarrow[R_2 \to \frac{1}{4} R_2]{\boxed{\text{scale}}} \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 8 & -1 & 6 \\ 0 & -7 & -2 & -3 \end{bmatrix}$$

$$\xrightarrow[R_4 \to R_4 + 7R_2]{R_3 \to R_3 - 8R_2} \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -17 & -2 \\ 0 & 0 & 12 & 4 \end{bmatrix} \xrightarrow[R_3 \to \left(\frac{-1}{17}\right) R_3, \ R_4 \to \left(\frac{1}{12}\right) R_4]{\boxed{\text{Scale}}} \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & \frac{2}{17} \\ 0 & 0 & 1 & \frac{1}{3} \end{bmatrix}$$

$$\xrightarrow{R_4 \to R_4 - R_3} \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & \frac{2}{17} \\ 0 & 0 & 0 & \underbrace{\frac{1}{3} - \frac{2}{17}}_{= \frac{11}{51}} \end{bmatrix}$$

$$\xrightarrow[R_4 \to \left(\frac{51}{11}\right) R_4]{\boxed{\text{Scale}}} \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & \frac{2}{17} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- \# Swaps $= 0$
- Scales : $\frac{1}{4} \cdot \frac{-1}{17} \cdot \frac{1}{12} \cdot \frac{51}{11}$

$$= \frac{-1}{16 \cdot 11} = \frac{-1}{176}$$

So $\boxed{\text{Det}(A) = -176}$.

(28.4) If A has a row consisting entirely of zeroes, then $\mathrm{Det}(A) = 0$.

Assume it is the i^{th} row. $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \hline 0 \quad 0 & \cdots & 0 \\ \hline a_{n1} & \cdots & a_{nn} \end{bmatrix} \leftarrow i^{th}$ row

Performing <u>Scale by 2 on i^{th} row</u> does not change A. but multiplies the determinant by 2.

$$\mathrm{Det}(A) = 2 \cdot \mathrm{Det}(A) \implies \mathrm{Det}(A) = 0.$$

(28.5) Let A_1 and A_2 be two $n \times n$ size matrices.

$$\boxed{\mathrm{Det}(A_1 \cdot A_2) = \mathrm{Det}(A_1) \cdot \mathrm{Det}(A_2)}.$$

<u>Main idea</u> If $A_1 \dashrightarrow B_1$ by an elementary row operation then $A_1 A_2 \dashrightarrow B_1 A_2$ by the same operation.

(proof of this is given in (28.6) below)

So, let B_1 be the reduced echelon form of A_1.

- Case 1. A_1 is not invertible. So $\mathrm{Det}(A_1) = 0$. and B_1 has a row consisting entirely of zeroes.

Means $B_1 A_2$ has a row consisting entirely of zeroes

$$\mathrm{Det}(A_1 A_2) = \begin{pmatrix}\text{Non-zero} \\ \text{scalar}\end{pmatrix} \cdot \mathrm{Det}(B_1 A_2) = 0 = \mathrm{Det}(A_1) \cdot \mathrm{Det}(A_2).$$

$$A_1 \xrightarrow[\substack{k \text{ swaps} \\ \ell \text{ scales by non-zero} \\ \text{scalars } c_1, c_2, \dots, c_\ell}]{} B_1 \text{ (reduced echelon form)}$$

If A_1 is invertible, then $B_1 = I_{n \times n}$; and

$$\mathrm{Det}(A_1) = \frac{(-1)^k}{c_1 c_2 \cdots c_\ell}.$$

By the "main idea" from the last page

$$A_1 A_2 \xrightarrow{\text{same row op's}} B_1 A_2 = I_{n\times n} A_2 = A_2.$$

So $\mathrm{Det}(A_1 A_2) = \dfrac{(-1)^k}{c_1 c_2 \cdots c_\ell} \cdot \mathrm{Det}(A_2) = \mathrm{Det}(A_1)\,\mathrm{Det}(A_2).$

(28.6)* Why does the "main idea" work?

Reason: row operations can be viewed as multiplying on the left by a particular matrix.

For instance, 2x2 case - $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix} \quad \text{(swap 1}^{\text{st}} \text{ \& 2}^{\text{nd}} \text{ rows)}$$

* Optional

$$\begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \alpha a & \alpha b \\ c & d \end{bmatrix}$$ (scale 1^{st} row by α)

$$\begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+\lambda c & b+\lambda d \\ c & d \end{bmatrix}$$ ($R_1 \to R_1 + \lambda R_2$ combine)

So, if $A_1 \dashrightarrow B_1$ via an elementary row operation

then $B_1 = R \cdot A_1$ for a matrix R

That means applying the same row operation to $A_1 A_2$

gives $A_1 A_2 \dashrightarrow R(A_1 A_2) = (RA_1) A_2$

$= B_1 A_2 .$