Lecture 29

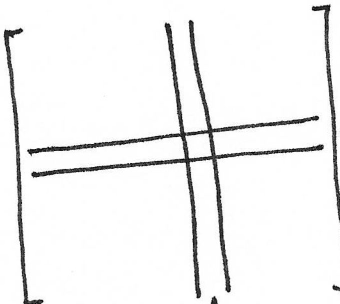
(29.0) Recall that we defined determinant of an $n \times n$ matrix in the following recursive manner.

- $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$

- if A is an $n \times n$ matrix, then for each (i,j) define

$$A_{ij} = (-1)^{i+j} \det$$

(cofactors)



delete i^{th} row

$(n-1) \times (n-1)$ size.

delete j^{th} column

- $$\boxed{\det(A) = a_{11} A_{11} + a_{12} A_{12} + \dots + a_{1n} A_{1n}}$$

(29.1) Last time we discussed how the determinant changes under row operations. The same is true for column operations:

- If B is obtained from A by one elementary row operation, then:

Row Operation	Effect
Swap: $R_i \leftrightarrow R_j$	$\det(B) = -\det(A)$
Scale: $R_i \to c \cdot R_i$	$\det(B) = c \det(A)$
Combine: $R_i \to R_i + \lambda R_j$	$\det(B) = \det(A)$

- If B is obtained from A by one elementary column operation, then

Column Operation	Effect
Swap $C_i \leftrightarrow C_j$	$\det(B) = -\det(A)$
Scale $C_i \to \alpha C_i$	$\det(B) = \alpha \det(A)$
Combine $C_i \to C_i + \lambda C_j$	$\det(B) = \det(A)$

- As det(triangular matrix) = product of diagonal entries, we can compute determinant of any matrix by bringing it to echelon form via row (or column) operations.

Since row operations for A = column operations for A^T, we conclude that

$$\boxed{\det(A) = \det(A^T)}.$$

(29.2) Summary of properties of the determinant.

- $\det(A) = 0$ if and only if A is not invertible

Meaning rows (or columns) of A are linearly dependent.

- $\boxed{\det(A_1)\det(A_2) = \det(A_1 A_2)}$

- $\boxed{\det(A) = \det(A^T)}$

- $\boxed{\det(I_n) = 1}$

For example: if A has <u>two repeated rows</u> then $\det(A) = 0$.
($R_i = R_j$ for a pair $i \neq j$)

because $R_i - R_j = 0$ is a linear dependence relation.

- $\det(A) = a_{i1} A_{i,1} + a_{i2} A_{i2} + \dots + a_{in} A_{in}$

for any i. Recall $A_{ij} = (-1)^{i+j} \cdot$ determinant of the $(n-1) \times (n-1)$ size matrix obtained from A by deleting i^{th} row and j^{th} column

$\boxed{\text{denote it by } M_{ij}}$

(Reason: $A = \begin{bmatrix} \cdots & R_1 & \cdots \\ \cdots & R_2 & \cdots \\ & \vdots & \\ \cdots & R_n & \cdots \end{bmatrix} \xrightarrow[\text{top. by Swaps}]{\text{bring } i^{th} \text{ row to the}} B = \begin{bmatrix} \cdots & R_i & \cdots \\ \cdots & R_1 & \cdots \\ \cdots & R_2 & \cdots \\ & \vdots & \\ \cdots & R_{i-1} & \cdots \\ \cdots & R_{i+1} & \cdots \\ & \vdots & \\ \cdots & R_n & \cdots \end{bmatrix}$

total $i-1$ swaps $\left\{ \begin{array}{l} i \longleftrightarrow i-1 \\ i-1 \longleftrightarrow i-2 \\ \vdots \\ 2 \longleftrightarrow 1 \end{array} \right.$

$\det(B) = (-1)^{i-1} \det(A)$. Use the definition for $\det(B)$

$$\det(B) = a_{i1} \cdot \det(M_{i1}) - a_{i2} \det(M_{i2}) + a_{i3} \det(M_{i3})$$
$$\cdots + (-1)^{n-1} a_{in} \det(M_{in})$$

So
$$\det(A) = (-1)^{i+1}\Big(a_{i1} \det(M_{i1}) - a_{i2} \det(M_{i2}) + \cdots\Big)$$
$$= a_{i1} A_{i1} + a_{i2} A_{i2} + \cdots + a_{in} A_{in}.$$

(29.3) Cofactor matrix and inverse of A. -

Given an $n \times n$ matrix A, define its <u>cofactor matrix</u>

$\mathrm{Cof}(A)$ (another $n \times n$ matrix) :

$$\mathrm{Cof}(A)_{ij} = (-1)^{i+j} \cdot \det(M_{ij})$$
$$= A_{ij}.$$

matrix obtained from A by removing i^{th} row & j^{th} col.

$$\boxed{A \cdot \mathrm{Cof}(A)^T = \det(A) \cdot I_n}$$

e.g. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Cofactors :
$$A_{11} = d$$
$$A_{12} = \cancel{b}\ -c$$
$$A_{21} = -b$$
$$A_{22} = a$$

$$A \cdot \mathrm{Cof}(A)^T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & -ab+ab \\ cd-dc & -bc+ad \end{bmatrix}$$
$$= (ad-bc) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$\boxed{A \cdot \mathrm{Cof}(A)^T = \det(A) \cdot I_{n \times n}}$$

(Proof. $(i,i)^{th}$ entry of $A \cdot \mathrm{Cof}(A)^T$ is :

$$a_{i1} A_{i1} + a_{i2} A_{i2} + \dots + a_{in} A_{in} = \det(A).$$

If $i \neq j$, $(i,j)^{th}$ entry of $A \cdot \mathrm{Cof}(A)^T$ is :

$$a_{i1} A_{j1} + a_{i2} A_{j2} + \dots + a_{in} A_{jn} \quad - (*)$$

Let B be $n \times n$ matrix with two repeated rows: i^{th} and j^{th} rows of B = i^{th} row of A. Every other row of B is same as that of A.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ & \vdots & & \\ a_{i1} & a_{i2} & \dots & a_{in} \\ & \vdots & & \\ a_{j1} & a_{j2} & \dots & a_{jn} \\ & \vdots & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \qquad B = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ & \vdots & & \\ a_{i1} & a_{i2} & \dots & a_{in} \\ & \vdots & & \\ a_{i1} & a_{i2} & \dots & a_{in} \\ & \vdots & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

i^{th} & j^{th} rows of B are equal

Note: $(*) = \det(B)$ (expanded along j^{th} row)

$= 0$ because rows of B are linearly dependent (see page 3 above). $\square$)

Hence if $\det(A) \neq 0$ (i.e. A is invertible) then

$$\boxed{A^{-1} = \frac{1}{\det(A)} \cdot \mathrm{Cof}(A)^T}$$

(29.4) Example. $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

Cofactors: $A_{11} = \det \begin{bmatrix} -1 & 1 \\ 2 & 2 \end{bmatrix} = -4$

$A_{12} = -\det \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = -3$

$A_{13} = \det \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} = 5$

$A_{21} = -\det \begin{bmatrix} 0 & 2 \\ 2 & 2 \end{bmatrix} = +4$; $A_{22} = \det \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} = 0$.

$A_{23} = -\det \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} = -2$; $A_{31} = \det \begin{bmatrix} 0 & 2 \\ -1 & 1 \end{bmatrix} = 2$.

$A_{32} = -\det \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = 3$; $A_{33} = \det \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} = -1$.

$\mathrm{Cof}(A) = \begin{bmatrix} -4 & -3 & 5 \\ 4 & 0 & -2 \\ 2 & 3 & -1 \end{bmatrix}$ $\quad \mathrm{Cof}(A)^T = \begin{bmatrix} -4 & 4 & 2 \\ -3 & 0 & 3 \\ 5 & -2 & -1 \end{bmatrix}$

$$A \cdot \mathrm{Cof}(A)^T = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 1 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} -4 & 4 & 2 \\ -3 & 0 & 3 \\ 5 & -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1(-4) + 0(-3) + 2(5) & 1(4) + 0(0) + 2(-2) & 1(2) + 0(3) + 2(-1) \\ 2(-4) + (-1)(-3) + 1(5) & 2(4) + (-1)(0) + 1(-2) & 2(2) + (-1)(3) + 1(-1) \\ 1(-4) + 2(-3) + 2(5) & 1(4) + 2(0) + 2(-2) & 1(2) + 2(3) + 2(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = 6 \cdot Id_{3\times 3}.$$

So, $$A^{-1} = \frac{1}{6} \cdot \begin{bmatrix} -4 & 4 & 2 \\ -3 & 0 & 3 \\ 5 & -2 & -1 \end{bmatrix}.$$

(29.5) General solution to a linear system whose coefficient matrix is square and invertible (Cramer's rule).

If $\boxed{A\vec{x} = \vec{b}}$ is a linear system of n equations in n variables

and A is invertible, then

$$\vec{x} = A^{-1} \cdot \vec{b} = \frac{1}{\det(A)} \cdot Cof(A)^T \cdot \vec{b}$$

Taking $(i,1)$ entry on both sides, we get

$$x_i = \frac{1}{\det(A)} \left(Cof(A)^T_{i1} b_1 + Cof(A)^T_{i2} b_2 + \dots + Cof(A)^T_{in} b_n \right)$$

$$\boxed{x_i = \frac{1}{\det(A)} \left(A_{1i} b_1 + A_{2i} b_2 + \dots + A_{ni} b_n \right)}$$

e.g. (same matrix as on page 6).

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 1 \\ 1 & 2 & 2 \end{bmatrix} \qquad \mathrm{Cof}(A) = \begin{bmatrix} -4 & -3 & 5 \\ 4 & 0 & -2 \\ 2 & 3 & -1 \end{bmatrix}$$

$$\det(A) = 6.$$

Consider the linear system $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$. Then

$$x_1 = \frac{1}{\det(A)} \left(A_{11} b_1 + A_{21} b_2 + A_{31} b_3 \right)$$

$$= \frac{1}{6} \left((-4)(2) + 4(2) + 2(4) \right) = \frac{8}{6} = \frac{4}{3}.$$

$$x_2 = \frac{1}{6} \left((-3)(2) + 0(2) + 3(4) \right) = 1.$$

$$x_3 = \frac{1}{6} \left(5(2) + (-2)(2) + (-1)(4) \right) = \frac{2}{6} = \frac{1}{3}.$$

Check. $$A\vec{x} = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 1 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 4/3 \\ 1 \\ 1/3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} + 0 + \frac{2}{3} \\ \frac{8}{3} - 1 + \frac{1}{3} \\ \frac{4}{3} + 2 + \frac{2}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} \checkmark$$