Frobenius and the group determinant

Sachin Gautam

Reading Classics. November 16, 2021

Main references

- Pavel Etingof et al. *Introduction to representation theory*, Student Mathematics Library volume 59, AMS publications (2010).
- Leonard Eugene Dickson *An elementary exposition of Frobenius' theory of group characters and group determinants*, Annals of Mathematics, second series, vol. 4, no. 1 (1902).
- Thomas Hawkins *The origins of the theory of group characters*, Archive for history of exact sciences, vol. 7, no. 2 (1971).
- mathshistory.st-andrews.ac.uk

Plan of the talk

- Georg Frobenius.
- Group determinant.
- Linear factors of the group determinant (Frobenius' Theorem 1).
- Irreducible factors of the group determinant (Frobenius' Theorem 2).
- Irreducible factors vs irreducible representations.
- Example of the dihedral group.

Georg Frobenius (1849-10-26 to 1917-08-03, Berlin)

- Joined University of Berlin in 1867. Studied under Kronecker, Kummer and Weierstraß.
- Obtained his doctorate in 1870 under the supervision of Weierstraß.
- Taught in Joachimsthal Gymnasium (his high school) 1870-1874.
- 1875-1892: Eidengnössische Polytechnikum, Zürich.
- Kronecker passed away in 1891. Frobenius got appointed Kronecker chair of mathematics in University of Berlin, 1892 (strong support from Fuchs and Weierstraß).

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Example. $G = \mathbb{Z}/2\mathbb{Z}$. Variables: x_0, x_1. $M_G(x_0, x_1) = \begin{bmatrix} x_0 & x_1 \\ x_1 & x_0 \end{bmatrix}$.

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Hence, $\Delta_G(x_0, x_1) = x_0^2 - x_1^2 = (x_0 + x_1)(x_0 - x_1)$.

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Example. $G = \mathbb{Z}/3\mathbb{Z}$. Variables: x_0, x_1, x_2.

$$M_G(x_0, x_1, x_2) = \begin{bmatrix} x_0 & x_1 & x_2 \\ x_2 & x_0 & x_1 \\ x_1 & x_2 & x_0 \end{bmatrix}$$

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$$\Delta_G(x_0, x_1, x_2) = (x_0 + x_1 + x_2)(x_0 + \omega_3 x_1 + \omega_3^2 x_2)(x_0 + \omega_3^2 x_1 + \omega_3 x_2)$$

where $\omega_3 = \exp\left(\frac{2\pi\iota}{3}\right)$.

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Example. $G = \mathbb{Z}/N\mathbb{Z}$ $(N \geq 2)$. Variables: $x_0, x_1, \ldots, x_{N-1}$.

$$M_{\mathbb{Z}/N\mathbb{Z}}(x_0, \ldots, x_{N-1}) = \begin{bmatrix} x_0 & x_1 & \cdots & x_{N-2} & x_{N-1} \\ x_{N-1} & x_0 & \cdots & x_{N-3} & x_{N-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_1 & x_2 & \cdots & x_{N-1} & x_0 \end{bmatrix}$$

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$$\Delta_{\mathbb{Z}/N\mathbb{Z}} = \prod_{k=0}^{N-1} \left(x_0 + \omega_N^k x_1 + \omega_N^{2k} x_2 + \cdots + \omega_N^{(N-1)k} x_{N-1} \right)$$

where $\omega_N = \exp\left(\frac{2\pi\iota}{N}\right)$.

Group determinant for abelian groups

Theorem (Dedekind)

If G is a finite abelian group, then

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- When $G = \mathbb{Z}/N\mathbb{Z}$ written as $\langle \sigma | \sigma^N = e \rangle$, there are exactly N group homomorphisms $\chi_k : G \to \mathbb{C}^\times$ $(0 \leq k \leq N-1)$, given by: $\chi_k(\sigma) = \omega_N^k$.

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- Using the structure theorem of finite abelian groups (Kronecker (1870)), it follows that for any finite abelian group G: $\left|\mathrm{Hom}_{\mathrm{gp}}(G, \mathbb{C}^\times)\right| = |G|$.

Dedekind-Frobenius correspondences, 1896

Dedekind wrote to Frobenius (March 25, 1896) a letter containing the definition of the group determinant and the factorization in the abelian case. Dedekind also hinted at some computations in the non-abelian case that he had done (without including them). Upon Frobenius' insistence, he hesitatingly formulated a conjecture in a letter dated April 3, 1896, for an arbitrary finite group G.

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(recall the commutator subgroup $[G, G]$ is the (normal) subgroup generated by $aba^{-1}b^{-1}$ for all $a, b \in G$). Dedekind ended the letter inviting Frobenius to pursue this conjecture:

I would be delighted if you wished to involve yourself with these matters, because I distinctly feel that I will not achieve anything here.

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Convention. Note that $\Delta_G(\underline{x})$ is homogeneous of degree $N = |G|$. Also, if x_e is the variable corresponding to the neutral element $e \in G$, then the coefficient of x_e^N in $\Delta_G(\underline{x})$ is 1. This is simply because the diagonal entries of $M_G(\underline{x})$ are all equal to x_e.
Here, and for the rest of this talk, a factor $p(\underline{x})$ of $\Delta_G(\underline{x})$ (necessarily homogeneous) will always assumed to be **monic with respect to the variable** x_e (that is, the coefficient of $x_e^{\deg(p)}$ is 1).

Frobenius' Theorem 1 (July, 1896)

Theorem (Frobenius)

Linear factors in $\Delta_G(\underline{x})$ are

$$\left\{ \ell_\chi(\underline{x}) = \sum_{g \in G} \chi(g) x_g \text{ where } \chi : G \to \mathbb{C}^\times \text{ is a group homomorphism} \right\}.$$

Moreover, each such factor appears with multiplicity 1.

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Remark

Note that, if $\chi : G \to \mathbb{C}^\times$ is a group homomorphism, then for every $a, b \in G$ we have: $\chi(aba^{-1}b^{-1}) = \chi(a)\chi(b)\chi(a)^{-1}\chi(b)^{-1} = 1$. Hence, $\chi([G, G]) = \{1\}$.

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$$\left|\mathrm{Hom}_{gp}(G, \mathbb{C}^\times)\right| = \left|\mathrm{Hom}_{gp}(G/[G, G], \mathbb{C}^\times)\right| = |G/[G, G]|$$

Proof of Frobenius' Theorem 1

Let $\chi : G \to \mathbb{C}^\times$ be a group homomorphism. $\ell_\chi(\underline{x}) := \sum_{g \in G} \chi(g) x_g$.

To prove: $\ell_\chi(\underline{x})$ divides $\Delta_G(\underline{x})$ with multiplicity 1.

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For $h \in G$, let Column(h) denote the h-th column of $M_G(\underline{x})$. Replace Column(e) by $\sum_{h \in G} \chi(h)$Column(h).

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$$M_G(\underline{x}) \rightsquigarrow \begin{bmatrix} \ell_\chi(\underline{x}) & * & \cdots & * \\ \vdots & * & \cdots & * \\ \chi(g)\ell_\chi(\underline{x}) & * & x_{g^{-1}h} & * \\ \vdots & * & \cdots & * \end{bmatrix}$$

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Because, g-th entry of Column(e) becomes:

$$\sum_{h\in G} \chi(h)x_{g^{-1}h} = \sum_{\sigma\in G} \chi(g\sigma)x_\sigma = \sum_{\sigma\in G} \chi(g)\chi(\sigma)x_\sigma = \chi(g)\ell_\chi(\underline{x}).$$

$\ell_\chi(\underline{x}) = \sum_{g\in G} \chi(g)x_g$ divides $\Delta_G(\underline{x})$ only once

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Row operation on A: Replace Row(g) by Row$(g) - \chi(g)$Row(e), for every $g \neq e$.

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$$A \rightsquigarrow \begin{bmatrix} 1 & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & * & \boxed{a_{g,h}} & * \\ 0 & * & \cdots & * \end{bmatrix}, \quad \begin{aligned} a_{g,h} &= x_{g^{-1}h} - \chi(g)x_h \\ &= \chi(gh^{-1})(\chi(g^{-1}h)x_{g^{-1}h} - \chi(h)x_h) \end{aligned}$$

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Hence, $\frac{\Delta_G(\underline{x})}{\ell_\chi(\underline{x})} = \mathrm{Det}(A)$ depends only on $\chi(a)x_a - \chi(b)x_b$.

$\ell_\chi(\underline{x}) = \sum_{g \in G} \chi(g) x_g$ divides $\Delta_G(\underline{x})$ only once

Fact. Let $P(w_a - w_b : 1 \leq a, b \leq n)$ be a (non-zero) polynomial in n variables, depending only on the differences of variables, as indicated. Then P is not divisible by $\sum_a w_a$.

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(For a proof of this fact, replace w_a by $w_a + \frac{t}{n}$. This does not change P, but adds t to $\sum_a w_a$. Assuming the contrary, we arrive at a linear polynomial in t dividing something independent of t, which is absurd.)

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The proof of this part is finished by taking $w_a = \chi(a) x_a$ ($a \in G$) and $P = \frac{\Delta_G(\underline{x})}{\ell_\chi(\underline{x})} = \mathrm{Det}(A)$.

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Brilliant idea. Consider three sets of variables $\underline{x} = \{x_g : g \in G\}$, $\underline{y} = \{y_g : g \in G\}$ and $\underline{z} = \{z_g : g \in G\}$ related by:

$$\underline{z} = \underline{x} * \underline{y} \text{ meaning } z_g = \sum_{\substack{a,b \in G \\ ab = g}} x_a y_b = \sum_{a \in G} x_a y_{a^{-1}g}.$$

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Then, $\boxed{M_G(\underline{z}) = M_G(\underline{x}) \cdot M_G(\underline{y})}$.

Proof. For $g, h \in G$, the (g,h)–th entry of $M_G(\underline{z})$ is given by:

$$z_{g^{-1}h} = \sum_{a \in G} x_a y_{a^{-1}g^{-1}h} = \sum_{c \in G} x_{g^{-1}c} y_{c^{-1}h} = \left(M_G(\underline{x}) M_G(\underline{y})\right)_{g,h}.$$

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$$\underline{z} = \underline{x} * \underline{y} \equiv \{z_g = \sum_a x_a y_{a^{-1}g}\}_{g \in G} \;\Rightarrow\; M_G(\underline{z}) = M_G(\underline{x}) M_G(\underline{y})$$

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Taking determinant, we get $\Delta_G(\underline{z}) = \Delta_G(\underline{x})\Delta_G(\underline{y})$.

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Note. Comparing coefficients of $x_a y_b$ on both sides, we get $\lambda_{ab} = \lambda_a \lambda_b$. That is, $g \mapsto \lambda_g$ is a group homomorphism, and $\ell = \ell_\lambda$ as desired.

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Put together, $\ell(\underline{x})\ell(\underline{y}) = c_1c_2\ell(\underline{z})$. But c_1c_2 is the coefficient of z_e in $\ell(\underline{z})$ assumed to be 1.

Frobenius' Theorem 2 (December 1896)

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Recall that conjugacy classes in G are equivalence classes under the equivalence relation: $a \sim b$ iff there exists g such that $a = gbg^{-1}$.

Example: $G = S_3$ symmetric group on 3 letters

Variables: $x_0, \ldots, x_5$ corresponding to the following ordering of permutations:

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Dedekind computed $\Delta_G = (u+v)(u-v)(u_1u_2 - v_1v_2)^2$, where ($\omega = \omega_3$ here):

$$\begin{aligned} u &= x_0 + x_1 + x_2, & v &= x_3 + x_4 + x_5, \\ u_1 &= x_0 + \omega x_1 + \omega^2 x_2, & v_1 &= x_3 + \omega x_4 + \omega^2 x_5, \\ u_2 &= x_0 + \omega^2 x_1 + \omega x_2, & v_2 &= x_3 + \omega^2 x_4 + \omega x_5. \end{aligned}$$

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Definitions. Let G be a group. In this talk, vector spaces are over $\mathbb{C}$.

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(Easy check: kernel and image of a G–intertwiner are subrepresentations of V and V' respectively.)

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Examples

Remark. (V, ρ) *is a G–representation* is same as saying $\rho : G \to \mathrm{GL}(V)$ is a group homomorphism. If $n = \dim(V)$, it is same as (after picking a basis of V) a group homomorphism $G \to \mathrm{GL}_n(\mathbb{C})$.

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Proof. Let (V, ρ) be a finite-dimensional representation. For every $g \in G$, there exists $m \in \mathbb{Z}_{\geq 1}$ such that $g^m = e$. So $\rho(g)^m = \mathrm{Id}_V$, hence $\rho(g)$ is diagonalizable. This implies that $\{\rho(g)\}_{g \in G}$ is a collection of pairwise commuting, diagonalizable matrices. Thus they can be diagonalized simultaneously, giving a joint eigenvector $0 \neq v \in V$. $\mathbb{C}v \subset V$ is a non-zero subrepresentation which will have to be equal to V, if V is irreducible.

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Note: $V_\zeta \cong V_{\zeta^{-1}}$. If $\zeta \neq \zeta^{-1}$, this representation is irreducible. If $\zeta = \pm 1$ (ζ could be -1 iff n is even), V_ζ further breaks into two 1–dimensional representations.

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PROOF. Any G–intertwiner $X : \mathbb{C}G \to V$ is completely determined by $v = X|e\rangle$. $(X|g\rangle = X(L(g)|e\rangle) = \rho(g)(X|e\rangle) = \rho(g)(v))$
Conversely, given $v \in V$, the map $|g\rangle \mapsto \rho(g)(v)$ is a G–intertwiner.
These assignments are inverse to each other and we are done.

Two fundamental results

Let G be a finite group. Let $\{(V_\lambda, \rho_\lambda) : \lambda \in \Lambda_G\}$ be the set of isomorphism classes of irreducible, finite-dimensional G–representations.

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Variables. $x_k \leftrightarrow r^k$ and $y_k \leftrightarrow sr^k$. Here $0 \leq k \leq n-1$.

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Danke Schön!