Lecture 1

Recap: Let k be a field. Let A be an algebra over k – i.e., A is a k-vector space, together with a bilinear, associative multiplication $A \times A \to A$ and a neutral element $1_A \in A$.

e.g. $A = k$, $k[x]$, $k[G]$ where G is a finite group,

$M_{n \times n}(k) = \mathrm{End}_k(V)$ (V is an n-dim'l k-vector space).

A representation of A is a vector space (over k) V, together with a homomorphism of k-algebras $\rho : A \to \mathrm{End}_k(V)$.

Notation: $A \circlearrowright V$ $\quad a \cdot v = \rho(a)(v) \; \forall a \in A, v \in V$.

Let V_1 and V_2 be two representations of A.

$$\mathrm{Hom}_A(V_1, V_2) := \{ f \in \mathrm{Hom}_k(V_1, V_2) \mid f(a \cdot x) = a \cdot f(x) \; \forall a \in A, x \in V_1 \}$$

$$\subset \mathrm{Hom}_k(V_1, V_2)$$

($\mathrm{Hom}_A(V_1, V_2)$ = vector space of all A-linear maps)

A subrepresentation of a representation V of A, is a vector subspace $U \subset V$ such that $a \cdot u \in U \; \forall a \in A, u \in U$.
(i.e. U is stable under the action by A)

We say V is irreducible if the only subrepresentations of V are $\{0\}$ and V.

A representation V of $\mathcal{A}$ is said to be indecomposable if $V \simeq V_1 \oplus V_2 \Rightarrow V_1 = \{0\}$ or $V_2 = \{0\}$. That is, V cannot be written as a direct sum of two non-zero subspaces, which are also sub-$\mathcal{A}$-representations.

§1. Schur's Lemma* - Let V, W be two representations of $\mathcal{A}$.

Let $f \in \operatorname{Hom}_{\mathcal{A}}(V, W)$.

(1) If V is irreducible, then either $f = 0$, or f is injective.

(2) If W is irreducible, then either $f = 0$, or f is surjective.

(3) Assume k is algebraically closed and $V = W$ is a finite-dim'l irreducible repn. of $\mathcal{A}$. Then there is $\lambda \in k$ such that $f = \lambda \cdot Id_V$.

Proof.- (1) $\operatorname{Ker}(f) \subset V$ is a subrepresentation. As V is irreducible, $\operatorname{Ker}(f) = \{0\}$ or V - i.e., f is either injective, or $f = 0$.

(2) - Same argument as in (1) for $\operatorname{Im}(f) \subset W$.

(3) $f : V \to V$ has an eigenvalue $\lambda \in k$ (since k is alg. closed)

Let $g = f - \lambda \cdot Id_V : V \to V$. g is again $\mathcal{A}$-linear and $\operatorname{Ker}(g) \neq \{0\}$ (eigenvectors of f with eigenvalue λ are in $\operatorname{Ker}(g)$).

Hence $g = 0$, i.e. $f = \lambda \cdot Id_V$. $\square$

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* Issai Schur (Jan. 10, 1875 - Jan. 10, 1941)

§2. Some consequences of Schur's lemma. – [k : alg. closed]

(a) Multiplicity of an irreducible repn : Assume $V = \bigoplus_{i=1}^{N} V_i^{\oplus d_i}$

where $V_1, \dots, V_N$ are pairwise non-isomorphic, finite-dim'l irreducible repns. Then,

$$d_j = \dim \operatorname{Hom}_{\mathcal{A}}(V_j, V) = \dim \operatorname{Hom}_{\mathcal{A}}(V, V_j)$$

(Proof. – $\operatorname{Hom}_{\mathcal{A}}(V_j, V) = \bigoplus_{i=1}^{N} \operatorname{Hom}_{\mathcal{A}}(V_j, V_i)^{\oplus d_i}$

By Schur's lemma, $\operatorname{Hom}_{\mathcal{A}}(V_j, V_i) = \begin{cases} \{0\} & \text{if } j \neq i, \\ k \cdot \mathrm{Id}_{V_i} & \text{if } j = i. \ \square \end{cases}$)

Hence $V \simeq \bigoplus_{i=1}^{N} V_i \otimes \operatorname{Hom}_{\mathcal{A}}(V_i, V)$.

(b) Let V be a f.d. irred. repn. of $\mathcal{A}$; and $n, m \in \mathbb{Z}_{\geq 1}$. Then

$$\operatorname{Hom}_{\mathcal{A}}(V^{\oplus n}, V^{\oplus m}) \simeq M_{m \times n}(k)$$

$$\varphi_X \longleftrightarrow X = (x_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$$

$$\varphi_X(v_1, \dots, v_n) = \left(\sum_{j=1}^{n} x_{ij} v_j \right)_{1 \leq i \leq m}$$

<u>Definition</u>. – A representation V of $\mathcal{A}$ is called <u>semisimple</u> (or, <u>completely reducible</u>) if V is isomorphic to a direct sum of irreducible repns.

§3. Theorem. Let $V_1, \dots, V_N$ be pairwise non-isomorphic, finite-dim'l, irreducible representations of A. Let $V = \bigoplus_{i=1}^{N} V_i^{\oplus n_i}$, and $W \subset V$ be a subrepresentation. Then W is semisimple – – i.e., $\exists\, r_i \in \{0, 1, \dots, n_i\}$, $1 \leq i \leq N$, s.t. $W \simeq \bigoplus_{i=1}^{N} V_i^{\oplus r_i}$.

[We are assuming k is algebraically closed.]

Proof. – Argue by induction on $\sum_{i=1}^{N} n_i$. Base case is trivial, so we proceed with carrying out the induction step. Assume, $W \neq \{0\}$.

– if W is irreducible, then for some projection

$$\begin{array}{c} V = \bigoplus_{i=1}^{N} V_i^{\oplus n_i} \\ \pi_j^{(s)} \downarrow \quad (1 \leq s \leq n_j) \\ V_j \end{array}$$

$$\pi_j^{(s)}\Big|_W : W \to V_j \quad \text{is non-zero}$$

$$\Big(\text{since } \pi_j^{(s)}(w) = 0 \ \ \forall\, \begin{array}{c} 1 \leq j \leq N \\ 1 \leq s \leq n_j \end{array} \Rightarrow w = 0\Big)$$

Schur's Lemma implies $W \simeq V_j$; and we are done

– if W is not irreducible. Let $P \subset W$ be an irreducible subrepn. of W (exists since W is finite-dim'l – so we can take P to be a non-zero (sub-)repn of W with smallest possible dimension). Same argument as above shows that $P \simeq V_j$ for some $1 \leq j \leq N$.

The inclusion $P \simeq V_j \subset W \subset V = \bigoplus_{i=1}^{N} V_i^{\oplus n_i}$ is given by

$$v \in V_j \longmapsto \begin{array}{ll} (\lambda_1 v, \dots, \lambda_{n_j} v) & \text{in } V_j^{\oplus n_j} \text{ piece} \\ 0 & \text{in } V_i^{\oplus n_i} \text{ piece } (i \neq j) \end{array}$$

(see §2 (a), (b) – above)

here, $\lambda_1, \ldots, \lambda_{n_j} \in k$ are not all zero.

Let $X \in GL_{n_j}(k)$ be such that $X \cdot \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_{n_j} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ and

let $\varphi_X \in \mathrm{Aut}_{\mathcal{A}}(V)$ given by $Id_{V_i^{\oplus n_i}}$ on i-th piece $(i \neq j)$

and $(v_1, \ldots, v_{n_j}) \longmapsto \left(\sum_{l=1}^{n_j} x_{pl} v_l \right)_{1 \leq p \leq n_j}$ on j-th piece.

Then,
$$\begin{array}{ccc} V_j & \hookrightarrow W \hookrightarrow V & \\ v & \longmapsto & (v, 0, \ldots, 0) \text{ in } V_j^{\oplus n_j}\text{-piece} \\ & & 0 \quad \text{ in } V_i^{\oplus n_i}\text{-piece} \quad i \neq j \end{array}$$

$$\Rightarrow W \simeq V_j \oplus K \quad \text{where} \quad K \subset V_1^{\oplus n_1} \oplus \cdots \oplus V_j^{\oplus n_j - 1} \oplus \cdots$$
$$= \mathrm{Ker}\left(W \hookrightarrow V \xrightarrow[\pi_j^{(1)}]{} V_j \right).$$

and we are done by induction. $\square$

Remark.- This theorem is valid for non-alg.-closed fields as well. Moreover, $V_1, \ldots, V_N$ do not have to be finite-dim'l. (see Remark 3.1.5 of [Etingof et al]). In this generality, $\mathrm{End}_{\mathcal{A}}(V)$ need not be k, but some division algebra (not necessarily commutative field) over k.

§4. <u>Corollary</u>. - [k - alg. closed]. Let V be a finite-dim'l irreducible $\mathcal{A}$-repn. Let $\{v_1, \dots, v_n\}$ be a set of <u>linearly independent</u> <u>vectors in V</u> and let $w_1, \dots, w_n \in V$ be arbitrary vectors. Then <u>$\exists\ a \in \mathcal{A}$ s.t. $a v_j = w_j\ \forall\ 1 \leq j \leq n$.</u>

<u>Proof</u>. Let $W = \{(a.v_1, \dots, a.v_n) \mid a \in \mathcal{A}\} \subset V^{\oplus n}$.

By Theorem $(1.3)^*$, $W \simeq V^{\oplus r}$ for some $0 \leq r \leq n$; and (by §2 above) the inclusion $W \subset V^{\oplus n}$ is given by an $n \times r$ matrix $X = (x_{ij}) \in M_{n\times r}(k)$

$$\varphi_X : V^{\oplus r} \hookrightarrow V^{\oplus n}$$

$$(w_1, \dots, w_r) \longmapsto \left(\sum_{j=1}^{r} x_{ij} w_j\right)_{1 \leq i \leq n}$$

If $r < n$, then there exists a non-zero $(q_1, \dots, q_n) \in k^n$ s.t.

$$(q_1, \dots, q_n)\, X = 0 \qquad \left(\text{i.e. } \sum_{i=1}^{n} q_i x_{ij} = 0\ \forall\ 1 \leq j \leq r\right)$$

more variables $q_1, \dots, q_n$ than eq^{n}s $\Rightarrow$ existence of a non-trivial soln.

As $(v_1, \dots, v_n) \in \mathrm{Im}(\varphi_X)$
$\exists\ u_1, \dots, u_r$ s.t.

$$v_1 = x_{11} u_1 + \dots + x_{1r} u_r$$
$$\vdots$$
$$v_n = x_{n1} u_1 + \dots + x_{nr} u_r$$

$$\Rightarrow \sum_{i=1}^{n} q_i v_i = \sum_{\substack{1 \leq i \leq n \\ 1 \leq j \leq r}} q_i x_{ij} u_j = 0$$

contradicting linear independence of $v_1, \dots, v_n$ $\square$

* Reference Thm N.M means Lecture N, §M