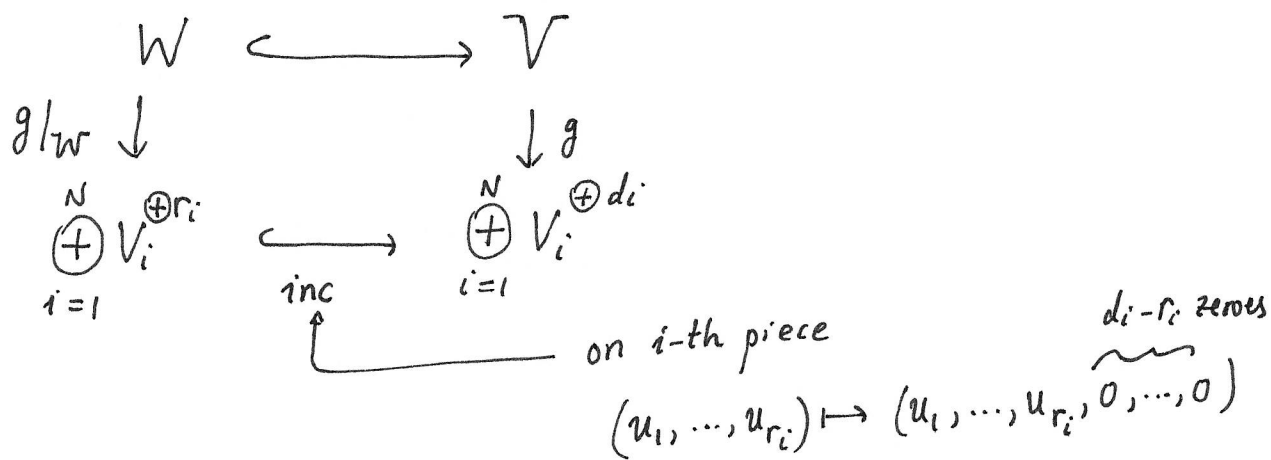


Recap - Theorem 3 of Lecture 1 - Let  $V_1, \dots, V_N$  be pairwise non-iso. finite-dimensional, irred. reps. of  $A$  (a unital, associative algebra over a field  $k$  - assumed to be alg. closed).

Let  $V = \bigoplus_{i=1}^N V_i^{\oplus d_i}$  ( $d_1, \dots, d_N \in \mathbb{Z}_{\geq 1}$ ). For any sub-repr.

$W \subset V$ , there exist  $r_1, \dots, r_N \in \mathbb{Z}_{\geq 0}$  ;  $r_j \leq d_j$ , and an automorphism  $g \in \text{Aut}_A(V)$  s.t.



§1. "Co-ordinate free" description.

Another way to think about the result stated above is to replace  $V^{\oplus d}$  by  $V \otimes D$  where  $D$  is a  $d$ -dim'l vector space over  $k$ .

Note - the identification  $V \otimes D \xrightarrow{\sim} \underbrace{V \oplus \dots \oplus V}_{d\text{-fold}}$  depends

on a choice of a basis  $\{\xi_1, \dots, \xi_d\}$  of  $D$

$$\sum_{i=1}^d u_i \otimes \xi_i \mapsto (u_1, u_2, \dots, u_d)$$

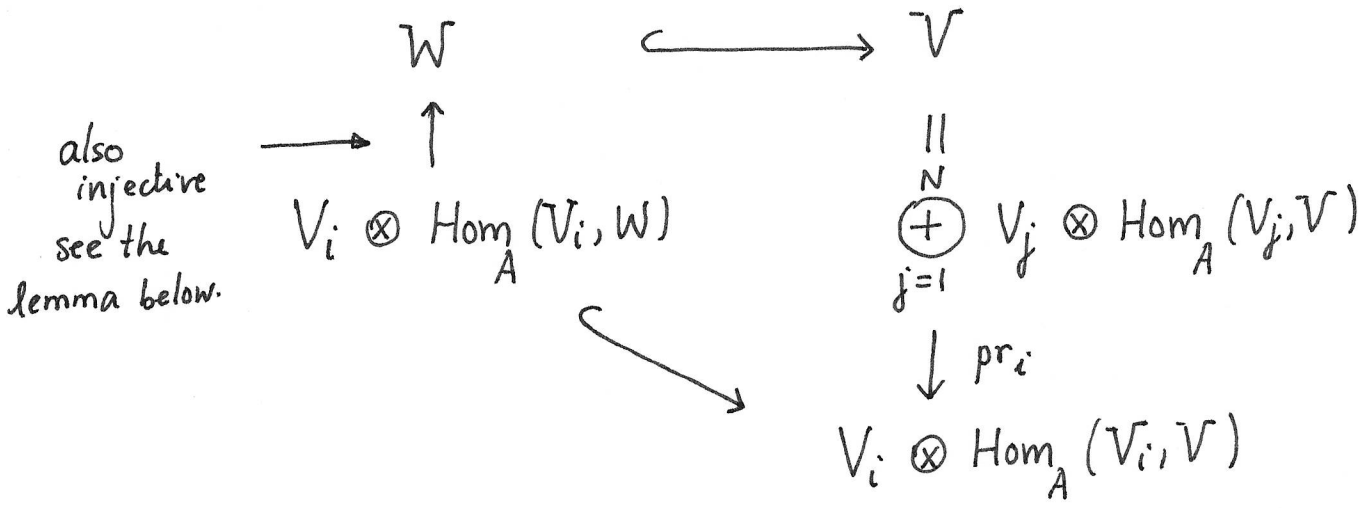
So, let  $V_1, \dots, V_N$  be as above and let  $D_1, \dots, D_N$  be finite-dimensional  $k$ -vectorspaces. (sometimes called auxiliary spaces or multiplicity spaces)

Let  $A \subset V_i \otimes D_i$  on the 1<sup>st</sup> tensor factor - i.e.,

$$a \cdot (v \otimes \xi) = (a \cdot v) \otimes \xi \quad \forall v \in V_i, \xi \in D_i.$$

$$V = \bigoplus_{i=1}^N V_i \otimes D_i. \quad \text{Remark - } D_i \cong \text{Hom}_A(V_i, V) \quad \left( \begin{array}{l} \text{see also} \\ \text{Lecture 1} \\ \text{\S 2 (a)} \end{array} \right)$$

Thus, given a subrepr.  $W \subset V$ , and  $1 \leq i \leq N$ , we get



Lemma. - Let  $U$  be a f.d., irred. repr. of  $A$ , and  $Y$  be a f.d. repr. Then the natural map

$$\alpha: U \otimes \text{Hom}_A(U, Y) \rightarrow Y \quad \text{is inj. (A-linear).}$$

$$u \otimes X \longmapsto X(u)$$

Proof. - Left as an exercise. -

Remark. - For any f.d. repr.  $Y$  of  $A$  and  $\{U_i\}_{i \in I}$  a finite collection of (f.d.) irr. reprs., we get an injective map

$$\text{(mutually non-iso)} \quad \bigoplus_{i \in I} U_i \otimes \text{Hom}_A(U_i, Y) \rightarrow Y.$$

$\gamma$  is called semisimple if this map is an iso.

§2. Density Theorem. - (see Corollary 4 of Lecture 1).

$V_1, \dots, V_N$  : pairwise non-iso., f.d. irred. reps. of  $A$ .

Theorem.  $\rho : A \longrightarrow \bigoplus_{j=1}^N \text{End}_k(V_j)$  is surjective.

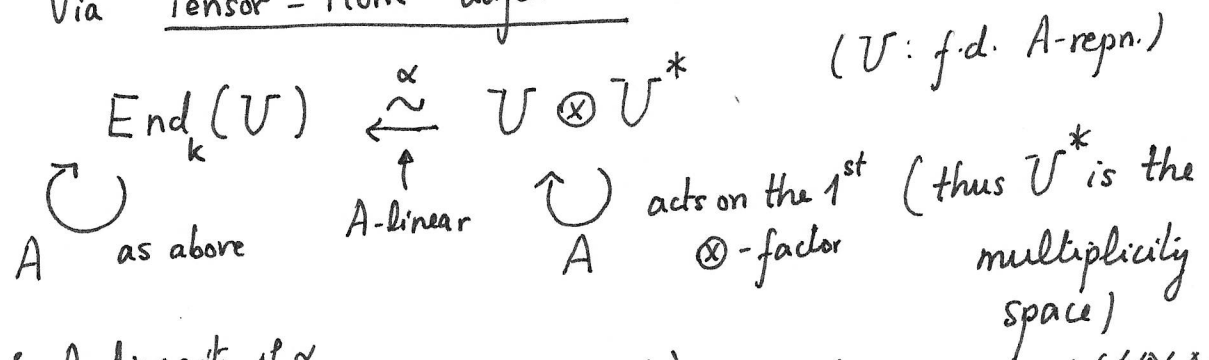
$$\rho(a) = (\rho_j(a))_{1 \leq j \leq N}$$

(for  $N=1$ , this theorem follows from Cor. 4 of Lecture 1).

Proof Let  $W = \text{Image}(\rho)$  and consider  $V = \bigoplus_{j=1}^N \text{End}_k(V_j)$  as an  $A$ -repn. via left multiplication

$$a \cdot (X_j)_{1 \leq j \leq N} = (\rho_j(a) X_j)_{1 \leq j \leq N}$$

Note - Via Tensor - Hom adjointness



(Proof of A-linearity of  $\alpha$ )

$$\alpha(a \cdot (u \otimes \xi)) = \alpha((a \cdot u) \otimes \xi) \neq \alpha(u \otimes (a \cdot \xi))$$

$$a \cdot (\alpha(u \otimes \xi)) = \rho(a) \circ \alpha(u \otimes \xi)$$

evaluate on  $\tilde{u} \in U$  :  $\alpha((a \cdot u) \otimes \xi)(\tilde{u}) = \xi(\tilde{u})(a \cdot u) \checkmark$

$$\rho(a)(\alpha(u \otimes \xi)(\tilde{u})) = a \cdot \left( \underbrace{\xi(\tilde{u})}_{\in k} u \right) = \xi(\tilde{u})(a \cdot u) \quad \square$$

So  $W \subset V = \bigoplus_{j=1}^N \text{End}(V_j) = \bigoplus_{j=1}^N V_j \otimes V_j^*$

$\Rightarrow W \simeq \bigoplus_{j=1}^N V_j \otimes L_j$  for some vector subspace  $L \subset V_j^*$ .

Assume  $L_j \subsetneq V_j^*$  for some  $j$ . Then  $\exists 0 \neq v \in V_j$  s.t.  
 $\eta(v) = 0 \quad \forall \eta \in L_j$ .

Hence,  $\text{Id}_{V_j} \otimes \text{ev}_v \circ \pi_j : V \rightarrow V_j \otimes V_j^* \rightarrow V_j$   
 restricted to  $W$  is identically zero.

But  $(\text{Id}_{V_i})_{1 \leq i \leq N} \in W = \text{Image}(\rho)$  and

$\text{Id}_{V_j} \otimes \text{ev}_v : V_j \otimes V_j^* \rightarrow V_j$   
 $\text{Id}_{V_j} \mapsto v \neq 0$  contradiction  $\square$

§3. Corollary ( $k$ : alg. closed) Assume  $\dim_k A < \infty$ . Then,

- (1) Every irred. repr. is finite-dim'l.
- (2) There are only finitely many (distinct) irred. reprs. of  $A$  -  
 $V_1, \dots, V_N$  and  $\sum_{j=1}^N \dim_k (V_j)^2 \leq \dim_k (A)$ .

Proof. - (1) follows since every non-zero vector in an irred. repr. generates it - and hence is a surjective image of  $A$  - finite-dim'l  
 (2) is a direct consequence of Thm above.  $\square$

§4. Example. -  $A = M_{n \times n}(k) \hookrightarrow V = k^n$

[a basis of  $A$  is given by elementary matrices -  $E_{ij}$  ( $1 \leq i, j \leq n$ )

Let  $|1\rangle, \dots, |n\rangle$  be the canonical basis of  $k^n$ .

Then  $E_{ij} |l\rangle = \delta_{jl} |i\rangle.$  ]

Homework exercise.  $V$  is an irred.  $A$ -representation.

By Corollary above. - as  $\dim(A) = \dim(V)^2$  - there are no other irreducible  $A$ -reps.

Proposition. - Let  $U$  be a f.d.  $A$ -repn. Then  $U \cong V^{\oplus r}$  for some  $r \in \mathbb{Z}_{\geq 0}$

(i.e. every f.d.  $A$ -repn is completely reducible)

( $r = \dim_A(V, U)$  by Schur's lemma)

Proof. - Since  $1_A = E_{11} + \dots + E_{nn}$  and

$$E_{ii}^2 = E_{ii} \quad ; \quad E_{ii} E_{jj} = 0 \text{ for } j \neq i; \text{ we}$$

get  $U \simeq \bigoplus_{i=1}^n E_{ii}(U)$  (as  $k$ -vector space).

Note:  $E_{ii}(U) \simeq E_{jj}(U)$  (convince yourself -

$$u \mapsto E_{ji}(u)$$

$$E_{ji}(u) \in E_{jj}(U) \quad \forall u \in U$$

$\Rightarrow$  for every non-zero  $u \in E_{11}(U)$

$$\text{Span} \{u, E_{21}(u), \dots, E_{n1}(u)\} \simeq k^n = V \text{ as } A\text{-reps}$$

(check).

Hence,  $U \simeq \underbrace{k^n \otimes_k E_{11}(U)}_{\substack{\text{auxiliary space} \\ r = \dim(E_{11}(U))}} \text{ as } A\text{-reps.}$   $\square$

§5. - Finite-dim'l semisimple algebras. - (Theorems of Artin & Wedderburn)

Let  $A$  be a finite-dim'l (as  $k$ -v.s.), algebra /  $k$ .

Let  $\{V_1, \dots, V_N\}$  = set of (iso-classes) all f.d. irred. reps. of  $A$ .

$\rho : A \rightarrow \bigoplus_{j=1}^N \text{End}(V_j)$  . Define: Radical of  $A$   
 $\text{Rad}(A) = \text{Ker}(\rho)$   
 (surjective by Thm. 3 above)

Theorem. - The following are equivalent:

- (1)  $\text{Rad}(A) = \{0\}$ .
- (2)  $A \simeq \bigoplus_{i=1}^N M_{d_i \times d_i}(k)$  ( $d_i = \dim V_i$ )
- (3) Every f.d. repr. of  $A$  is semisimple.
- (4)  $A \hookrightarrow A$  by left mult. is semisimple.

In this case, we say  $A$  is a (f.d.) semisimple algebra.

Proof. - (1)  $\Leftrightarrow$  (2) by Density theorem (see §2 above).

(3)  $\Rightarrow$  (4) since  $A$  is finite-dim'l

(2)  $\Rightarrow$  (3): Let  $A_i = \text{End}_k(V_i) \hookrightarrow V_i$  (the only irred. repr. of  $A_i$  - see Prop. §4 above.)

(7)

As  $A \simeq A_1 \oplus \dots \oplus A_N$   $1_A = (1_{A_1}, \dots, 1_{A_N})$   
 (as  $k$ -algebras)

Let  $U$  be a f.d. repr. of  $A$ . Then  $U \simeq \bigoplus_{j=1}^N 1_{A_j}(U)$ .

Moreover  $A \subset 1_{A_j}(U)$  through  $j$ -th projection:

$$(a_1, \dots, a_N) \cdot (1_{A_j}(u)) = a_j(u).$$

Hence (as  $A_j$  is semisimple w/ only one irred. repr  $V_j$ ):

$$U \simeq \bigoplus_{j=1}^N V_j^{\oplus d_j} \quad \text{for some } d_1, \dots, d_N \in \mathbb{Z}_{\geq 0}.$$

(4)  $\Rightarrow$  (2). If  $A \subset A$  is a semisimple repr, then by Schur's

lemma:  $A \simeq \bigoplus_{j=1}^N V_j \otimes \underbrace{D_j}_{\text{(mult. space)}}$

where  $D_j \simeq \text{Hom}_A(V_j, A) \cong \text{Hom}_A(A, V_j) \ni \varphi$   
 $\simeq V_j \ni \varphi(1)$

So  $\dim(A) = \sum_{j=1}^N (\dim V_j)^2$  and hence the natural map

$$A \rightarrow \bigoplus_{j=1}^N \text{End}_k(V_j) \text{ is an iso.} \quad \square$$