Lecture 2

Recap - Theorem 3 of Lecture 1 - Let $V_1, \ldots, V_N$ be pairwise non-iso. finite-dimensional, irred. repns. of A (a unital, associative algebra over a field k - assumed to be alg. closed).

Let $V = \bigoplus_{i=1}^{N} V_i^{\oplus d_i}$ $(d_1, \ldots, d_N \in \mathbb{Z}_{\geq 1})$. For any sub-repn. $W \subset V$, there exist $r_1, \ldots, r_N \in \mathbb{Z}_{\geq 0}$; $r_j \leq d_j$, and an automorphism $g \in \mathrm{Aut}_A(V)$ s.t.

$$\begin{array}{ccc} W & \hookrightarrow & V \\ g|_W \downarrow & & \downarrow g \\ \bigoplus_{i=1}^{N} V_i^{\oplus r_i} & \xrightarrow[\text{inc}]{} & \bigoplus_{i=1}^{N} V_i^{\oplus d_i} \end{array}$$

on i-th piece

$$(u_1, \ldots, u_{r_i}) \mapsto (u_1, \ldots, u_{r_i}, \underbrace{0, \ldots, 0}_{d_i - r_i \text{ zeroes}})$$

§1. "Co-ordinate free" description.

Another way to think about the result stated above is to replace $U^{\oplus d}$ by $U \otimes D$ where D is a d-dim'l vector space over k.

Note - the identification $U \otimes D \xrightarrow{\sim} \underbrace{U \oplus \cdots \oplus U}_{d\text{-fold}}$ depends on a choice of a basis $\{\xi_1, \ldots, \xi_d\}$ of D

$$\sum_{i=1}^{d} u_i \otimes \xi_i \mapsto (u_1, u_2, \ldots, u_d)$$

So, let $V_1, \dots, V_N$ be as above and let $D_1, \dots, D_N$ be finite-dimensional k-vectorspaces. (sometimes called auxiliary spaces or multiplicity spaces)

Let $A \circlearrowright V_i \otimes \underline{D_i}$ on the 1^{st} tensor factor – i.e.,

$$a \cdot (v \otimes \xi) = (a \cdot v) \otimes \xi \quad \forall\, v \in V_i,\ \xi \in D_i .$$

$V = \bigoplus_{i=1}^{N} V_i \otimes D_i$. $\underline{\text{Remark}}$ – $D_i \underset{\text{as k-v.s.'s}}{\simeq} \mathrm{Hom}_A(V_i, V)$ (see also Lecture 1 §2 (a))

Thus, given a subrepn. $W \subset V$, and $1 \leq i \leq N$, we get

also injective see the lemma below.

$$\begin{tikzcd}
W \arrow[r, hook] & V \arrow[d, equal] \\
V_i \otimes \mathrm{Hom}_A(V_i, W) \arrow[u] \arrow[dr, hook] & \bigoplus_{j=1}^{N} V_j \otimes \mathrm{Hom}_A(V_j, V) \arrow[d, "pr_i"] \\
 & V_i \otimes \mathrm{Hom}_A(V_i, V)
\end{tikzcd}$$

Lemma. – Let U be a f.d., irred. repn. of A, and Y be a f.d. repn. Then the natural map

$$\alpha : U \otimes \mathrm{Hom}_A(U, Y) \longrightarrow Y \quad \text{is inj. (A-linear).}$$
$$u \otimes X \longmapsto X(u)$$

Proof. – Left as an exercise. –

Remark. – For any f.d. repn. Y of A and $\{U_i\}_{i \in I}$ a finite collection of (f.d.) irr. repns. (mutually non-iso), we get an injective map

$$\bigoplus_{i \in I} U_i \otimes \mathrm{Hom}_A(U_i, Y) \to Y.$$

Y is called semisimple if this map is an iso.

§2. Density Theorem.– (see Corollary 4 of Lecture 1).

$V_1, \dots, V_N$: pairwise non-iso., f.d. irred. repns. of A.

<u>Theorem.</u> $\rho : A \longrightarrow \bigoplus_{j=1}^{N} \mathrm{End}_k(V_j)$ is surjective.

$$\rho(a) = (\rho_j(a))_{1 \le j \le N}$$

(for $N=1$, this theorem follows from Cor. 4 of Lecture 1).

<u>Proof</u> Let $W = \mathrm{Image}(\rho)$ and consider $V = \bigoplus_{j=1}^{N} \mathrm{End}_k(V_j)$ as an A-repn. via left multiplication

$$a \cdot (X_j)_{1 \le j \le N} = (\rho_j(a) X_j)_{1 \le j \le N}$$

<u>Note</u> – Via <u>Tensor – Hom adjointness</u>

$$\mathrm{End}_k(U) \xleftarrow[\sim]{\alpha} U \otimes U^* \qquad (U : \text{f.d. } A\text{-repn.})$$

$A \circlearrowright \mathrm{End}_k(U)$ as above; α : A-linear; $A \circlearrowright U \otimes U^*$ acts on the 1st $\otimes$-factor (thus U^* is the multiplicity space)

(<u>Proof of A-linearity of α</u>
$$\alpha(a \cdot (u \otimes \xi)) = \alpha((a \cdot u) \otimes \xi) \ \cancel{= \xi(u \cdot a)}$$
$$a \cdot (\alpha(u \otimes \xi)) = \rho(a) \circ \alpha(u \otimes \xi)$$

evaluate on $\tilde{u} \in U$:
$$\alpha((a \cdot u) \otimes \xi)(\tilde{u}) = \xi(\tilde{u})\,(a \cdot u) \quad \checkmark$$
$$\rho(a)\big(\alpha(u \otimes \xi)(\tilde{u})\big) = a \cdot (\underbrace{\xi(\tilde{u})}_{\in k}\, u) = \xi(\tilde{u})(a \cdot u) \quad \square \Big)$$

So $\quad W \subset V = \bigoplus_{j=1}^{N} \mathrm{End}(V_j) = \bigoplus_{j=1}^{N} V_j \otimes V_j^*$

$$\Rightarrow \quad W \simeq \bigoplus_{j=1}^{N} V_j \otimes L_j \quad \text{for some vector subspace } L \subset V_j^*.$$

Assume $L_j \subsetneq V_j^*$ for some j. Then $\exists\ 0 \neq v \in V_j$ s.t.

$$\eta(v) = 0 \quad \forall\, \eta \in L_j.$$

Hence, $\quad \mathrm{Id}_{V_j} \otimes ev_v \circ \pi_j : V \longrightarrow V_j \otimes V_j^* \to V_j$

restricted to W is identically zero.

But $\quad (\mathrm{Id}_{V_i})_{1 \leq i \leq N} \in W = \mathrm{Image}(\rho)$ and

$$\mathrm{Id}_{V_j} \otimes ev_v : V_j \otimes V_j^* \to V_j$$
$$\mathrm{Id}_{V_j} \longmapsto v \neq 0 \quad \text{contradiction} \qquad \square$$

§3. Corollary (k: alg. closed) Assume $\dim_{k\text{-v.s.}} A < \infty$. Then,

(1) Every irred. repn. is finite-dim'l.

(2) There are only finitely many (distinct) irred. repns. of A - $V_1, \ldots, V_N$ and $\sum_{j=1}^{N} \dim_k (V_j)^2 \leq \dim_{k\text{-v.s.}} (A)$.

Proof.- (1) follows since every non-zero vector in an irred. repn. generates it - and hence is a surjective image of A ← finite-dim'l $\square$

(2) is a direct consequence of Thm above.

§4. Example. – $A = M_{n\times n}(k) \curvearrowright V = k^n$

[a basis of A is given by elementary matrices – E_{ij} $(1\le i,j\le n)$

Let $|1\rangle, \dots, |n\rangle$ be the canonical basis of k^n.

Then $E_{ij}\,|l\rangle = \delta_{jl}\,|i\rangle$.]

Homework exercise. V is an irred. A-representation.

By Corollary above. – as $\dim(A) = \dim(V)^2$ – there are no other irreducible A-repns.

<u>Proposition</u>. – Let U be a f.d. A-repn. Then $U \cong V^{\oplus r}$ for some $r \in \mathbb{Z}_{\ge 0}$

(i.e. <u>every f.d. A-repn is completely reducible</u>) $(r = \dim_A(V,U)$ by Schur's lemma)

<u>Proof</u>. – Since $1_A = E_{11} + \dots + E_{nn}$ and

$E_{ii}^2 = E_{ii}$; $E_{ii}E_{jj} = 0$ for $j \ne i$; we

get $U \simeq \bigoplus_{i=1}^{n} E_{ii}(U)$ (as k-vector space).

<u>Note</u>: $E_{ii}(U) \xrightarrow{\sim} E_{jj}(U)$ (convince yourself –

$u \longmapsto E_{ji}(u)$ $\qquad E_{j1}(u) \in E_{jj}(U)$ $\forall u \in U$)

$\Rightarrow$ for every non-zero $u \in E_{11}(U)$

$\mathrm{Span}\{u, E_{21}(u), \dots, E_{n1}(u)\} \simeq k^n = V$ as A-repns

(Check).

Hence, $\quad V \simeq \Bbbk^n \otimes_{\Bbbk} \underbrace{E_{11}(V)}$ as A-repns.

auxiliary space

$r = \dim(E_{11}(V))$ $\square$

§5. - Finite-dim'l semisimple algebras. - (Theorems of Artin & Wedderburn)

Let A be a finite-dim'l (as $\Bbbk$-v.s.), algebra / $\Bbbk$.

Let $\{V_1, \dots, V_N\}$ = set of (iso-classes reps.) all f.d. irred. repns. of A.

$$\rho : A \to \bigoplus_{j=1}^{N} \mathrm{End}(V_j)$$

(surjective by Thm. 3 above)

. Define: Radical of A

$$\mathrm{Rad}(A) = \mathrm{Ker}(\rho)$$

Theorem. — The following are equivalent:

(1) $\mathrm{Rad}(A) = \{0\}$.

(2) $A \simeq \bigoplus_{i=1}^{N} M_{d_i \times d_i}(\Bbbk)$ $\quad (d_i = \dim V_i)$

(3) Every f.d. repn. of A is semisimple.

(4) $A \circlearrowright A$ by left mult. is semisimple

In this case, we say A is a (f.d.) semisimple algebra.

Proof. — (1) $\Leftrightarrow$ (2) by Density theorem (see §2 above).

(3) $\Rightarrow$ (4) since A is finite-dim'l

(2) $\Rightarrow$ (3) : Let $A_i = \mathrm{End}_{\Bbbk}(V_i) \circlearrowright V_i$ (the only irred. repn. of A_i - see Prop. §4 above)

As $A \simeq A_1 \oplus \cdots \oplus A_N$ (as k-algebras) $\qquad 1_A = (1_{A_1}, \ldots, 1_{A_N})$

Let U be a f.d. repn. of A. Then $U \simeq \bigoplus_{j=1}^{N} 1_{A_j}(U)$.

Moreover $A \circlearrowright 1_{A_j}(U)$ through j-th projection:

$$(a_1, \ldots, a_N) \cdot (1_{A_j}(u)) = a_j(u).$$

Hence (as A_j is semisimple w/ only one irred. repn V_j):

$$U \simeq \bigoplus_{j=1}^{N} V_j^{\oplus d_j} \quad \text{for some } d_1, \ldots, d_N \in \mathbb{Z}_{\geq 0}.$$

(4) $\Rightarrow$ (2). If $A \circlearrowright A$ is a semisimple repn, then by Schur's lemma:

$$A \simeq \bigoplus_{j=1}^{N} V_j \otimes \underset{\uparrow}{D_j} \quad \text{(mult. space)}$$

where
$$D_j \simeq \mathrm{Hom}_A(V_j, A) \simeq \mathrm{Hom}_A(A, V_j) \ni \varphi$$
$$\simeq V_j \ni \varphi(1_A)$$

So $\dim(A) = \sum_{j=1}^{N} (\dim V_j)^2$ and hence the natural map

$A \to \bigoplus_{j=1}^{N} \mathrm{End}_k(V_j)$ is an iso. $\square$