Lecture 3

Let A be an algebra over a field k. (We will not assume that k is alg. closed)

§1. Theorem (Krull-Schmidt) Let V be a finite-dimensional A-repn. Then $V \simeq V_1 \oplus \dots \oplus V_n$ where $V_1, \dots, V_n$ are indecomposable repns. This decomposition is unique upto reordering of terms.

The proof of this theorem rests on the following lemma. (often obtained as a corollary of "Fitting's lemma".

§2 Lemma. Let U be a finite-dim'l, indecomposable repn. of A. Let $\varphi \in \mathrm{End}_A(U)$. Then φ is either an isomorphism, or φ is nilpotent.

Proof. Consider the chains of A-subrepns. of U:

$$\mathrm{Ker}(\varphi) \subset \mathrm{Ker}(\varphi^2) \subset \cdots$$
$$\mathrm{Im}(\varphi) \supset \mathrm{Im}(\varphi^2) \supset \cdots$$

(must stabilize since $\dim U < \infty$)

Let N be such that
$$\mathrm{Ker}(\varphi^N) = \mathrm{Ker}(\varphi^{N+j}) \quad \forall j \geq 0$$
$$\mathrm{Im}(\varphi^N) = \mathrm{Im}(\varphi^{N+j}) \quad \forall j \geq 0$$

Let $K = \mathrm{Ker}(\varphi^N)$, $I = \mathrm{Im}(\varphi^N)$. Claim: $U \simeq K \oplus I$.

Given the claim, indecomposability of U implies that either $K = U$ (i.e. $\varphi^N = 0$) or $I = U$ (i.e. φ^N is an iso., hence φ is an iso.)

<u>Proof of the claim.</u> – Let $u \in U$. Since $\mathrm{Im}(\varphi^N) = \mathrm{Im}(\varphi^{2N})$,

$\exists\, w \in U$ s.t. $\varphi^N(u) = \varphi^{2N}(w)$.

$$\Rightarrow \quad u = \underbrace{\left(u - \varphi^N(w)\right)}_{\text{in } \mathrm{Ker}(\varphi^N)} + \underbrace{\varphi^N(w)}_{\text{in } \mathrm{Im}(\varphi^N)}.$$

So, $U = K + I$. It remains to check that $K \cap I = \{0\}$.

Let $u \in K \cap I$, i.e. $\varphi^N(u) = 0$ and $u = \varphi^N(v)$ for some $v \in U$.

So, $\varphi^{2N}(v) = \varphi^N(u) = 0 \Rightarrow v \in \mathrm{Ker}(\varphi^{2N}) = \mathrm{Ker}(\varphi^N)$

Hence, $\varphi^N(v) = (u =)\ 0$. $\square$

§3. <u>Proof of Krull-Schmidt Theorem.</u> –

Existence of the decomposition follows by an easy induction argument on $\dim V$: if V is indecomposable, then there is nothing to prove.

otherwise $V \simeq V_1 \oplus V_2$ and $\dim V_j < \dim V$ $(j = 1, 2)$.

By induction, V_1 and V_2 can be written as a finite direct sum of indecomposables, hence the same is true for V.

<u>Uniqueness</u>: Assume $V \simeq V_1 \oplus \cdots \oplus V_n \simeq W_1 \oplus \cdots \oplus W_m$

where $V_1, \ldots, V_n$ and $W_1, \ldots, W_m$ are indecomposable.

Consider $\theta_j \in \mathrm{End}_A(V_1)$ defined as:

$(1 \leq j \leq m)$

$$\begin{tikzcd}
V_1 \arrow[r, hook, "i_1"] \arrow[dd, "\theta_j"'] & \bigoplus\limits_{s=1}^{n} V_s \arrow[r, "g", "\cong"'] & \bigoplus\limits_{t=1}^{m} W_t \arrow[d, "\pi'_j"] \\
 & & W_j \arrow[d, "i'_j"] \\
V_1 & \bigoplus\limits_{s=1}^{n} V_s \arrow[l, "\pi_1"'] & \bigoplus\limits_{t=1}^{m} W_t \arrow[l, "g^{-1}"']
\end{tikzcd}$$

(i & π are canonical inclusions & projections resp. for V; i', π' for W)

$$\theta_j = \pi_1 \circ g^{-1} \circ i_j \circ \pi_j \circ g \circ i_1 .$$

Since $\sum\limits_{j=1}^{m} i_j \circ \pi_j = Id_W$, $\left(W = \bigoplus\limits_{j=1}^{m} W_j\right)$ $\quad \sum\limits_{j=1}^{m} \theta_j = \pi_1 \circ i_1 = Id_{V_1}$.

By lemma (§2) above, each θ_j is either nilpotent or an iso.

<u>Claim</u>: Sum of nilpotent elements from $End_A(V_1)$ is again nilpotent. (Proof in §4 below).

Assuming the claim, $\theta_1 + \ldots + \theta_m = Id \Rightarrow$ there is some θ_j - by relabelling, assume $(j=1)$ θ_1 is an iso.

Hence $\pi_1 \circ g \circ i_1 : V_1 \to W_1$ is an iso, with inverse $\theta_1^{-1} \circ \pi_1 \circ g^{-1} \circ i_1$. Identifying these two indecomposables, we get

$$V_1 \oplus \left(\bigoplus_{s=2}^{n} V_s\right) \xrightarrow[\cong]{g} V_1 \oplus \left(\bigoplus_{t=2}^{m} W_t\right)$$

Let $B = \bigoplus_{s=2}^{n} V_s$ and $B' = \bigoplus_{t=2}^{m} W_t$. Let $h : B \to B'$ be given

by:

$$\begin{tikzcd} B \arrow[r, hook] \arrow[rrr, bend right, "h"'] & V_1 \oplus B \arrow[r, "\sim", "g"'] & V_1 \oplus B' \arrow[r] & B' \end{tikzcd}$$

. Then $Ker(h) = \{0\}$

from identification $g(x,y) = (x,z)$.

(if $h(v) = 0$, then $g(0, v) = (0,0) \Rightarrow v = 0$.)

Thus, by dimension reasons, $B \simeq B'$ and we are done by induction. $\square$

§4. <u>Lemma</u>. Let $\varphi_1, \dots, \varphi_n \in End_A(U)$ (U: f.d. indec. A-rep.) be nilpotent. Then $\varphi_1 + \dots + \varphi_n$ is nilpotent.

<u>Proof</u>. - (Induction on n - base case $n = 1$ is obvious).

If $\varphi = \varphi_1 + \dots + \varphi_n$ is not nilpotent, then it is invertible (by Lemma §2).

$$\Rightarrow Id_U = \varphi^{-1}\varphi_1 + \dots + \varphi^{-1}\varphi_n.$$

$$\Rightarrow Id_U - \varphi^{-1}\varphi_n = \varphi^{-1}\varphi_1 + \dots + \varphi^{-1}\varphi_{n-1}.$$

Since φ_j is nilpotent, $\varphi^{-1}\varphi_j$ cannot be invertible, hence is nilpotent $\forall\, 1 \leq j \leq n$.

By induction, $\sum_{j=1}^{n-1} \varphi^{-1}\varphi_j$ is nilpotent. But

$Id - \varphi^{-1}\varphi_n$ is invertible ($(1-X)^{-1} = 1 + X + \dots + X^{N-1}$ if $X^N = 0$)

This contradiction proves that φ is not invertible, hence is nilpotent $\square$