A: algebra over a field k. Recall that we proved in Lecture 3, §3 -

- Krull-Schmidt Theorem. - Every finite-dimensional repn. of A can be written as a finite direct sum of indecomposable repns, uniquely (up to reordering of the direct summands)

Missing step in the proof (see Lecture 3- pages 2,3 for the set up):

If $V_1 \oplus \dots \oplus V_n \xrightarrow[g]{\sim} W_1 \oplus \dots \oplus W_m$ ($V_1, \dots, V_n$ & $W_1, \dots, W_m$ are f.d. indecomposables)

then $\exists\, j \in \{1, \dots, m\}$ s.t.

$\theta_j : V_1 \to V_1$ is an isomorphism.

$$\begin{tikzcd}
V_1 \arrow[r, hook, "i_1"] \arrow[ddd, "\theta_j"', "\cong"] & \bigoplus_{s=1}^{n} V_s \arrow[r, "g"] & \bigoplus_{t=1}^{m} W_t \arrow[d, "\pi_j'"] \\
 & & W_j \arrow[d, "i_j'"] \\
 & & \bigoplus_{t=1}^{m} W_t \arrow[d, equal] \\
V_1 & \bigoplus_{s=1}^{n} V_s \arrow[l, "\pi_1"'] & \bigoplus_{t=1}^{m} W_t \arrow[l, "g^{-1}"']
\end{tikzcd}$$

→ This implies $V_1 \xrightarrow{\sim} W_j$

Proof (missing in Lecture 3)

$$V_1 \xrightarrow[a]{\pi_j' \circ g \circ i_1} W_j \xrightarrow[b]{\pi_1 \circ g^{-1} \circ i_j'} V_1 \qquad b \circ a = \theta_j \text{ is an iso}$$

$\Rightarrow$ $W_j \simeq \mathrm{Im}(a) \oplus \mathrm{Ker}(b)$. Since $\forall\, w \in W_j$, $\exists\, v \in V_1$ s.t. $b(w) = \theta_j(v)$

So $w = \underbrace{w - a(\cancel{\theta_j}(v))}_{\mathrm{Ker}(b) \text{ since}} + \underbrace{a(\cancel{\theta_j}(v))}_{\text{in } \mathrm{Im}(a)} \Rightarrow W_j = \mathrm{Im}(a) + \mathrm{Ker}(b)$

$$b(w) - b(a(v)) = b(w) - \theta_j(v) = 0$$

Also, $w \in \mathrm{Im}(a) \cap \mathrm{Ker}(b) \Rightarrow w = a(v)$ for some $v \in V_1$ and $0 = b(w) = b(a(v)) = \theta_j(v) \Rightarrow v = 0$ (since θ_j is iso.)

As a is injective, $\mathrm{Im}(a) \neq \{0\}$ $(V_1 \neq \{0\})$ $\Rightarrow$ (by indecomposability of W_j), $W_j \simeq \mathrm{Im}(a)$

hence $a : V_1 \to W_j$ is an iso. $\square$

§1. Jordan - Hölder Series. Given a finite-dimensional repn. V of A, an easy induction on dimension shows that we can find a chain of sub-A-repns

$$(*) \left[\begin{array}{l} V = U_0 \supsetneq U_1 \quad \cdots \quad \supsetneq U_n \supsetneq U_{n+1} = \{0\} \quad \text{s.t.} \\ \qquad U_j / U_{j+1} \text{ is irreducible } \forall\, 0 \leq j \leq n. \\ \qquad \text{(called irr. factors of the J-H series)} \end{array}\right.$$

[use the argument as in Krull. Schmidt Thm - Lecture 3, page 2.]

Any such chain of subrepns. of V is called a Jordan- Hölder series (or filtration).

Theorem. Assume $V = U_0 \supsetneq U_1 \supsetneq \cdots \supsetneq U_{n+1} = \{0\}$ are
$V = W_0 \supsetneq W_1 \supsetneq \cdots \supsetneq W_{m+1} = \{0\}$

two J-H filtrations of an A-repn. V. Let $Y_i = U_i / U_{i+1}$, $Z_j = W_j / W_{j+1}$ $\left(\begin{array}{l} 0 \leq i \leq n \\ 0 \leq j \leq m \end{array}\right)$.

Then $m = n$ and $\exists\, \sigma \in S_{n+1}$ (permutation) s.t. $Y_i \simeq Z_{\sigma(i)}$ $\forall\, 0 \leq i \leq n$.

That is, Jordan-Hölder filtration need not be uniquely determined, but its irred. factors are.

Proof.- Assume $n \leq m$ and argue by induction on ~~$n = \min\{m,n\}$~~ say, $m+n$.

(Base case: $n = 0$ means V is irreducible

Hence $W_1 \subsetneq V$ has to be zero - so, $m=1$ and

$W_0 = V = U_0$.)

Let $U = U_n$ and $W = W_m$ (two irred. subrepns. of V).

As $U \cap W$ is a subrepn of U (& W), we get: either $U \cap W = U = W$ or $U \cap W = \{0\}$.

If $U = W$, we get two J-H filtrations of $V/U = \overline{V}$:

$$\overline{V} = U_0/U_n \supsetneq U_1/U_n \supsetneq \cdots \supsetneq U_{n-1}/U_n \supsetneq \{0\}$$

$$= W_0/W_m \supsetneq W_1/W_m \supsetneq \cdots \supsetneq W_{m-1}/W_m \supsetneq \{0\}$$

and we are done by induction.

Otherwise, $U \oplus W \subset V$ and we let $\overline{V} = V/_{U \oplus W}$.

Let $\overline{V} = P_0 \supsetneq \cdots \supsetneq P_t \supsetneq \{0\}$ be a J-H filtration with irred. factors $X_\ell = P_\ell / P_{\ell+1}$ $(0 \leq \ell \leq t)$. Here, for instance, we can take $\{\pi(U_i)\}_{i=0}^{n}$ $(\pi : V \to \overline{V})$ and eliminate repeated terms to get J-H series for $\overline{V}$, so $t < n$.

By induction hypothesis we can conclude that:

- V/W has 2 J-H series $\quad \pi_1^{-1}(P_0) \supsetneq \cdots \supsetneq \pi_1^{-1}(P_t) \supsetneq U \supsetneq \{0\}$

  $\left(\pi_1 : V/W \to V/U\oplus W\right) \quad W_0/W \supsetneq \cdots \supsetneq W_{m-1}/W_m \supsetneq \{0\}.$

  Hence, $t = m-2$ and $\{X_0, \ldots, X_{m-2}, U\} = \{Z_0, \ldots, Z_{m-1}\}$ (with multiplicities)

- V/U has 2 J-H Series: $\quad \pi_2^{-1}(P_0) \supsetneq \cdots \supsetneq \pi_2^{-1}(P_t) \supsetneq W \supsetneq \{0\}$

  $\left(\pi_2 : V/U \to V/U\oplus W\right) \quad U_0/U \supsetneq \cdots \supsetneq U_{n-1}/U \supsetneq \{0\}.$

  Hence, $t = n-2$ and $\{X_0, \ldots, X_{m-2}, W\} = \{Y_0, \ldots, Y_{n-1}\}$ (with mult.)

Combining we get $m = n$ and

$$\{Y_0, \ldots, Y_{n-1}, Y_n = U\} = \{Z_0, \ldots, Z_{m-1}, W\}$$

(with multiplicities). $\square$

§2. Summary of main theorems:

Given A: an algebra over k, [often coming from number theory, geometry, combinatorics, or physics]

⇝ $\mathrm{Rep}(A)$: category of A-repns. (Objects: repns. of A ; Morphisms: A-linear maps)

$\cup$

$\mathrm{Rep}_{fd}(A)$: category of finite-dimensional repns. of A.

(object of study).

Let $\mathrm{Rep}^{ss.}_{fd}(A) \subset \mathrm{Rep}_{fd}(A)$ be the (full) subcategory, consisting of finite-dimensional, semisimple repns.

Main questions: (1) <u>Classify irreducible, finite-dim'l A-repns.</u>

$$\mathrm{Irr}_{fd}(A) := \text{set of iso. classes of f.d. irr. repns. of } A$$
$$= \{V_\lambda\}_{\lambda \in P(A)}$$
(some indexing set $P(A)$ - depending on A).
- just a notation.

<u>Schur's Lemma</u>.- • $\mathrm{Hom}_A(V_\lambda, V_\mu) = \{0\}$ if $\lambda \neq \mu$.

• $f \in \mathrm{End}_A(V_\lambda) \Rightarrow f = 0$ or f is invertible.

If k is alg. closed: • $\mathrm{End}_A(V_\lambda) = k \cdot \mathrm{Id}_{V_\lambda}$.

• for every $V \in \mathrm{Rep}^{ss}_{fd}(A)$, $V \simeq \bigoplus_{\lambda \in P(A)} V_\lambda^{d_\lambda(V)}$, where

$$d_\lambda(V) = \dim \mathrm{Hom}_A(V_\lambda, V) = \dim \mathrm{Hom}_A(V, V_\lambda).$$

• For $V, W \in \mathrm{Rep}^{ss}_{fd}(A)$,

$$\mathrm{Hom}_A(V, W) = \bigoplus_{\lambda \in P(A)} \mathrm{Mat}_{d_\lambda(W) \times d_\lambda(V)}(k)$$

(ie, if we manage to answer question (1) above - i.e., identify the set $P(A)$ - we completely understand the category $\mathrm{Rep}^{ss}_{fd}(A)$.)

<u>Jacobson's density Theorem</u> :- If $\lambda_1, \dots, \lambda_n \in P(A)$ (i.e. $V_{\lambda_1}, \dots, V_{\lambda_n}$ are distinct, irred, f.d. A-repns.), then

$$A \to \bigoplus_{i=1}^{n} \mathrm{End}_k(V_{\lambda_i}) \text{ is surjective}$$

Hence
$$\sum_{i=1}^{n} \dim(V_{\lambda_i})^2 \leq \dim_{k\text{-v.s.}}(A)$$
(if A is f.d. as k-vector space)

Question (2) - Compute dimensions of f.d. irred. repns. of A - Give explicit construction of V_λ $\forall\, \lambda \in P(A)$.

Question (3) - Is $\mathrm{Rep}^{ss}_{fd}(A) = \mathrm{Rep}_{fd}(A)$? (Complete reducibility question).

- If the answer to question (3) is NO, then -

Question (4) - Classify indecomposable repns. of A (same problems as above - for indec. repns.)

$$\mathrm{Indec}_{fd}(A) = \text{set of iso. classes of f.d. indecomposable repns. of } A$$

$$= \bigcup_{\ell=0}^{\infty} \underbrace{\mathrm{Indec}^{(\ell)}_{fd}(A)}_{\substack{\text{those indec. with } \ell = \text{ length of} \\ \text{J-H. series (see §1 above)}}} \quad [\text{disjoint union}]$$

(Knull-Schmidt - Every $V \in \mathrm{Rep}_{fd}(A)$ is of the form : $\bigoplus_{r=1}^{N} U_r$ ($U_1, \ldots, U_r$ are indec. repns. of A).

§3. Extensions - One possible approach towards answering question (4) above would be to proceed with increasing length.

$$\mathrm{Indec}^{(1)}_{fd}(A) = \mathrm{Irr}_{fd}(A).$$

$\mathrm{Indec}^{(2)}_{fd}(A)$ consists of repns (indec.) V s.t $\exists\, \overset{\text{irr.}}{V_1} \subset V$ s.t V/V_1 is again irred.

– i.e., V fits in a non-split short exact sequence (s.e.s.):

$$0 \to V_1 \to V \to V_2 \to 0.$$

This is one of the main motivations for studying <u>Ext</u> (short for extensions)

$$\mathrm{Ext}^1_A(V_2, V_1) = \frac{\text{set of s.e.s.'s } \quad 0 \to V_1 \to V \to V_2 \to 0}{\text{equivalence rel}^{n} \text{ defined below}}$$

Let $\xi : 0 \to V_1 \to V \to V_2 \to 0$ be two s.e.s.'s. We say $\xi \sim \eta$ if
$\eta : 0 \to V_1 \to W \to V_2 \to 0$ $\exists$ isomorphisms making each square commute:

$$\begin{array}{ccccccccc} 0 & \to & V_1 & \to & V & \to & V_2 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & V_1 & \to & W & \to & V_2 & \to & 0 \end{array}$$

[see Problem 10 for more details.]