# Lecture 5
## Representations of groups.

Let $G$ be a group. A <u>representation</u> of $G$ (over a field $k$) is the data of a $k$-vector space $V$, together with a group hom. $\rho_V : G \to GL(V)$

Thus, $G$ acts on $V$ via invertible linear operators

$$g \cdot v = \rho_V(g)(v) \quad \forall\, g \in G,\ v \in V$$

Notation: $G \overset{\curvearrowright}{\underset{\rho_V}{}} V$ ← often omitted.

When $G$ is a finite group, one can define a unital assoc algebra $k[G]$ over $k$ so that the notion of $G$-repn coincides with that of $k[G]$-repns.

## §1. Group algebra of a finite group $G$.

Let $k[G]$ be a vector space over $k$ with basis $\{\delta_g : g \in G\}$.

Define $* : k[G] \times k[G] \to k[G]$ on basis vectors as:

$$\delta_g * \delta_h = \delta_{gh} \quad \forall\, g, h \in G$$

and extend by bilinearity - i.e., $\left(\sum_{g\in G} c_g \delta_g\right) * \left(\sum_{h \in G} d_h \delta_h\right)$

$$= \sum_{g,h \in G} c_g d_h \delta_{gh}$$

$$= \sum_{g \in G} \Big(\sum c_{g_1} d_{g_2}\Big) \delta_g$$

over $g_1, g_2 \in G$ s.t. $g_1 \cdot g_2 = g$

(This product is often called <u>convolution product</u>)

$k[G], *, \delta_e =$ unit

is the group algebra of $G$ over $k$.

It is often useful (for generalization purposes) to view $k[G]$ as the $k$-vector space of functions $\{G \to k\}$ :

$\sum_{g \in G} c_g \delta_g \; : \; h \mapsto c_h$ . In this form, the convolution product becomes :

$$(f_1 * f_2)(g) = \sum_{\substack{g_1, g_2 \in G : \\ g_1 g_2 = g}} f_1(g_1) f_2(g_2)$$

§2. <u>G-repns. = $k[G]$-repns.</u> (G: finite gp. ; $k$: field)

Given $\alpha : G \to GL(V)$ a group hom., we get

$\rho : k[G] \to \mathrm{End}_k(V)$ via: $\rho\left(\sum_{g \in G} c_g \delta_g\right) = \sum_{g \in G} c_g \, \alpha(g)$

(check: $\rho$ is an alg. hom)

Conversely, given an alg. hom. $\rho : k[G] \to \mathrm{End}_k(V)$, $\rho(\delta_e) = \mathrm{Id}_V$ and $\delta_g * \delta_{g^{-1}} = \delta_{g^{-1}} * \delta_g = \delta_e$, we get that $g \mapsto \rho(\delta_g) \in GL(V)$. It is easy to see that this assignment is a group hom.

§3. <u>Reviewing terminology from $k[G]$-repn.</u> : Given a group $G$ and G-repns. $V, W$ (over $k$), define

(i) $\mathrm{Hom}_G(V, W) = \{ \underset{k\text{-linear}}{f : V \to W} \mid f(g \cdot v) = g \cdot f(v) \;\; \forall g \in G, v \in V \}$

$\subset \mathrm{Hom}_k(V, W)$ [G-linear maps or G-intertwiners]

(ii) $G \circlearrowright V \oplus W$ via $g \cdot (v, w) = (g \cdot v, g \cdot v)$ $\forall g \in G, v \in V, w \in W$.

i.e. - in matrix form:

$$\rho_{V \oplus W}(g) = \left[\begin{array}{c|c} \rho_V(g) & 0 \\ \hline 0 & \rho_W(g) \end{array}\right]$$

- we have absolutely similar notions of irreducible / indec. repns., and analogue of Schur's Lemma etc.

<u>Schur's Lemma</u> If $G \circlearrowright V, W$ are two irreducible repns. and $\varphi \in \mathrm{Hom}_G(V, W)$ then either $\varphi = 0$ or an isomorphism.

($\dim V < \infty$; $k$ : alg. closed) $\mathrm{End}_G(V) = k \cdot \mathrm{Id}_V$.

§4. <u>Maschke's Theorem.</u> Assume $G$ is a finite group and characteristic of $k$ does not divide ~~$|G|$~~ $|G|$ (i.e., $|G| \neq 0$ in $k$).

Then every finite-dim'l $G$-repn. is semisimple.

Proof. - Let $G \circlearrowright V$ be a f.d. $G$-repn. Given a non-zero, proper subrepn. $U \subset V$, we will prove the existence of a (complementary) subrepn. $W \subset V$ s.t. $V = U \oplus W$. This shows that $V$ can be written as a direct sum of irred. subrepns - hence is semisimple (see Problem 3. - we don't have to assume $k$ is alg. closed there.)

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Heinrich Maschke Oct. 24, 1853 - March 1, 1908

Let $W' \subset V$ be s.t. $V = U \oplus W'$ as $k$-vector spaces.

Let $\pi : V \to U$ be the natural projection. $\pi$ is not necessarily a $G$-intertwiner - since $W'$ was chosen arbitrarily. However, we can "average" it - define $P : V \to U$ as

$$P(v) = \frac{1}{|G|} \sum_{g \in G} g(\pi(g^{-1}(v)))$$

( ie. $P = \frac{1}{|G|} \sum_{g \in G} \rho(g) \circ \pi \circ \rho(g)^{-1}$ $(\rho : G \to GL(V))$ ).

Claim: $P|_U = Id_U$ and $P$ is a $G$-intertwiner.

Proof: The first assertion follows from $U$ being a sub-repn, and $\pi|_U = Id_U$. For the second, let $g \in G$, $v \in V$:

$$P(g \cdot v) = \frac{1}{|G|} \sum_{\sigma \in G} \cancel{\sigma(\pi(\sigma^{-1} g^{-1}(}\ \sigma \pi \sigma^{-1}(g \cdot v))$$

$\sigma^{-1} g = h^{-1}$
so, $\sigma = gh$.

$$= \frac{1}{|G|} \sum_{h \in G} g h \pi h^{-1}(v)$$

$$= g \cdot P(v).$$

Let $W = \mathrm{Ker}(P) \subset V$ subrepn. Then $V = U \oplus W$ since:

- for $v \in V$, $v = \underbrace{(v - P(v))}_{\text{in Ker}(P)} + \underbrace{P(v)}_{\text{in } U}$.
- $u \in U \cap W \Rightarrow u = P(u) = 0$.

□

## §5. Corollaries of Maschke's Theorem + Schur's Lemma.-

Assume $G$ is a finite group and $k$ is an alg. closed field s.t. char$(k)$ does not divide $|G|$.

(i) $k[G]$ is a semisimple algebra. Hence there are only finitely many f.d. irred $G$-repns : $\{V_\lambda : \lambda \in P(G)\}$ ( $P(G)$ - finite indexing set).

and
$$\boxed{\sum_{\lambda \in P(G)} \dim(V_\lambda)^2 = \dim_{k\text{-v.s.}} k[G] = |G|}$$

<u>Another (direct) proof.</u> Consider $G \circlearrowright k[G]$ via left mult.

$$\mathcal{L}(g)(\delta_h) = \delta_{gh} \quad \forall g \in G, h \in G$$

$$\mathcal{L} : G \to GL(k[G]).$$

By Maschke's Thm.
$$k[G] \simeq \bigoplus_{\lambda \in P(G)} V_\lambda^{\oplus d_\lambda}$$
direct sum of f.d. irred. $G$-repns.

where, by Schur's lemma, $d_\lambda = \dim \operatorname{Hom}_G(k[G], V_\lambda)$.

Now $\operatorname{Hom}_G(k[G], X) \simeq X$ for any $G$-repn. $X$ via

$$f \longmapsto f(\delta_e)$$

( injectivity: if $f(\delta_e) = 0$, then $f(\delta_g) = f(\mathcal{L}_g(\delta_e))$
$= g \cdot f(\delta_e) = 0$ ; i.e. $f = 0$.

surjectivity: given $x \in X$ define $f : k[G] \to X$ (so $f(\delta_e) = x$).
$f(\delta_g) = g \cdot x$
is clearly a $G$-intertwiner )

$\Rightarrow d_\lambda = \dim(V_\lambda)$ and hence $|G| = \sum_{\lambda \in P(G)} (\dim V_\lambda)^2$. In particular $|P(G)| < \infty$. $\square$

Remark.- Maschke's Theorem fails if char(k) divides $|G|$ :

e.g. $\mathbb{Z}/p\mathbb{Z} \to GL_2(\mathbb{F}_p)$ ← or any field of char $= p$.

$x \mapsto \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$ is indecomposable, but not irreducible.

(see also Prop. 4.1.2 of Pasha's book.)

## §6. Tensor and dual of G-repns.

[These operations on representations are not available for repns. of an arbitrary algebra over k.]

Let $G$ be a group.

Let $V$ and $W$ be two repns. of $G$ over a field $k$.

- We have a natural $G$-action on $V \otimes W$ given by

$$g \cdot (v \otimes w) = (g \cdot v) \otimes (g \cdot w) \quad \forall\, g \in G,\ v \in V,\ w \in W.$$

Heading for confusion. Recall that for $A$-repn $U$ and an "auxiliary space" $D$
$(A: \mathrm{alg}/k)$

we earlier considered $A \circlearrowright U \otimes D$ by $a \cdot (u \otimes \xi) = (a \cdot u) \otimes \xi$

according to which $U \otimes D \simeq U^{\oplus \dim(D)}$.

To avoid confusion, we will underline the auxiliary space - if such action is being considered for $k[G]$ :

→ $G \circlearrowright V \otimes \underline{W}$ means $g \cdot (v \otimes w) = (g \cdot v) \otimes w$
↑ "aux. space"

Thus $V \otimes \underline{W} \cong V^{\oplus \dim(W)}$ as $G$-repns.

→ $G \circlearrowright V \otimes W$ will always be the action defined here: $(g \cdot (v \otimes w) = (g \cdot v) \otimes (g \cdot w)$.

- $G \circlearrowright \mathrm{Hom}_k(V,W)$ by $g\cdot(X) = \rho_W(g)\circ X\circ \rho_V(g)^{-1}$

$$(\forall g\in G,\ X\in \mathrm{Hom}_k(V,W))$$

(Special case : $W = k$ with trivial $G$-action: $g\cdot\lambda = \lambda\ \forall g\in G, \lambda\in k$)

$$\mathrm{Hom}_k(V,k) = V^*$$

- $G \circlearrowright V^*$ by $(g\cdot\xi)(v) = \xi(g^{-1}\cdot v)\quad \forall g\in G, \xi\in V^*, v\in V.$

<u>Remark.</u> - for $G \circlearrowright V, W$, two repns. of $G$, the most natural repns. to define are $V\otimes W$ and $\mathrm{Hom}_k(V,W)$ as <u>$G\times G$-repns</u>:

$$(g,h)\cdot(v\otimes w) = (g\cdot v)\otimes(h\cdot w)\qquad (g,h\in G;\ v\in V, w\in W)$$

$$((g,h)\cdot X)(v) = h\cdot(X\cdot(g^{-1}\cdot v))\qquad \begin{pmatrix} X\in \mathrm{Hom}_k(V,W)\\ v\in V\end{pmatrix}$$

The way we have viewed these vector spaces as <u>$G$-repns.</u> above, is via the diagonal embedding $G \hookrightarrow G\times G \circlearrowright V\otimes W$ or $\mathrm{Hom}_k(V,W)$

## §7. <u>Tensor-Hom adjointness map is $G\times G$-linear.</u>

Let $V, W$ be two $G$-repns. Let $\alpha: V^*\otimes W \to \mathrm{Hom}_k(V,W)$ be the linear map $\alpha(\xi\otimes w): v\mapsto \xi(v)\,w\quad \forall \xi\in V^*, w\in W, v\in V.$

<u>Lemma</u> (left as an easy exercise) $\alpha$ is $G\times G$-intertwiner.