Lecture 6

Let G be a finite group and let k be an alg. closed field s.t. $\mathrm{char}(k)$ does not divide $|G|$. $\quad k[G]$ = group algebra of G over k.

§1. Recall that we proved: (see Lecture 5, §4.5)

- Every f.d. repn. of G over k is semisimple (Maschke's Thm.)
- $$k[G] \simeq \bigoplus_{\lambda \in P(G)} V_\lambda^{\oplus \dim V_\lambda}.$$ Here, $P(G)$ is an indexing set

labelling irred. f.d. repns. of G $\quad$ ($\mathrm{Irr}_{fd}(G) = \{ V_\lambda : \lambda \in P(G) \}$)

$$|G| = \sum_{\lambda \in P(G)} (\dim V_\lambda)^2$$

<u>Cor.</u> $\psi : k[G] \longrightarrow \bigoplus_{\lambda \in P(G)} \mathrm{End}(V_\lambda)$ is an iso. of algebras.

$$\sum_{g \in G} c_g \delta_g \longmapsto \Big(\sum_{g \in G} c_g \alpha_\lambda(g) \Big)_{\lambda \in P(G)}$$

if $\alpha_\lambda : G \to GL(V_\lambda)$ is the action hom.

Proof. As each $k[G] \to \mathrm{End}(V_\lambda)$ is an alg. hom., so is ψ.

We will prove that ψ is injective, and the corollary will follow from dimensional equality. So, if $\psi\Big(\sum_{g \in G} c_g \delta_g\Big) = 0$, then the element $\xi = \sum_{g \in G} c_g \delta_g$ acts as 0 on each irred. f.d. repn. of G, and hence by Maschke's Thm, ξ acts as 0 on every f.d. repn. of G.

Considering the action $G \underset{\mathcal{L}}{\circlearrowright} k[G]$ (via left mult. $g \cdot \delta_h = \delta_{gh}$)

we get $0 = \mathcal{L}(\xi)(\delta_e) = \sum_{g \in G} c_g \, \mathcal{L}(g)(\delta_e) = \sum_{g \in G} c_g \, \delta_g = \xi$. $\square$

§2. Symmetries of the isomorphism

$$\psi : k[G] \longrightarrow \bigoplus_{\lambda \in P(G)} \operatorname{End}_k(V_\lambda)$$

- Let $G \times G \circlearrowright k[G]$ via left - right mult.

$$(g_1, g_2) \cdot \delta_g = \delta_{g_1 g g_2^{-1}}$$

- $\forall \lambda \in P(G)$, $G \times G \circlearrowright \operatorname{End}(V_\lambda)$ via:

$$((g_1, g_2) \cdot X)(v) = \alpha_\lambda(g_1)(X(\alpha_\lambda(g_2)^{-1} \cdot v))$$

Then ψ is a $G \times G$ - intertwiner.

Proof. - It is enough to verify the statement for a particular $\lambda \in P(G)$ - i.e. $\psi_\lambda : k[G] \to \operatorname{End}(V_\lambda)$ is $G \times G$ - intertwiner.

<u>To prove</u>: $\psi_\lambda((g_1, g_2) \cdot \delta_\sigma) = (g_1, g_2) \cdot \psi_\lambda(\delta_\sigma) \quad \forall g_1, g_2, \sigma \in G$.

$$\text{Left-hand side} = \psi_\lambda(\delta_{g_1 \sigma g_2^{-1}}) = \alpha_\lambda(g_1 \sigma g_2^{-1})$$

$$= \alpha_\lambda(g_1) \cdot \alpha_\lambda(\sigma) \cdot \alpha_\lambda(g_2)^{-1}$$

$$= (g_1, g_2) \cdot \psi_\lambda(\delta_\sigma) = \text{Right-hand side}$$

$\square$

§3. Corollary of $G\times G$-intertwining property of $\psi : k[G] \to \bigoplus_{\lambda\in P(G)} \operatorname{End} V_\lambda$

Consider $G \circlearrowright k[G]$ via conjugation – i.e. $G \to G\times G \to \underset{k\text{-v.s.}}{\overset{\mathrm{Aut}}{\cancel{\operatorname{End}}}}(k[G])$

$$C(g)\,(\delta_\sigma) = \delta_{g\sigma g^{-1}} \quad \forall\, g\in G,\ \sigma\in G.$$

And analogously, $\underset{\text{conj.}}{G \circlearrowright} \operatorname{End}(V_\lambda)$: $C(g)(X) = \alpha_\lambda(g)\, X\, \alpha_\lambda(g)^{-1}$
$\forall\, g\in G,\ X\in \operatorname{End}(V_\lambda)$.

Then via ψ : $\underset{(\text{conj. action})}{k[G]^G} \xrightarrow{\sim} \bigoplus_{\lambda\in P(G)} \underset{(\text{conj. action})}{\operatorname{End}(V_\lambda)^G} = \bigoplus_{\lambda\in P(G)} \operatorname{End}_G(V_\lambda)$

$$= \bigoplus_{\lambda\in P(G)} k\cdot \mathrm{Id}_{V_\lambda}$$

[Notation- for $G\circlearrowright V$, $V^G = \{v\in V \mid g\cdot v = v\ \forall g\in G\}$.]

Note : $k[G]^G = \{ f : G\to k \mid f(g\sigma g^{-1}) = f(\sigma)\ \forall\, g,\sigma\in G\}$

i.e., functions constant on conjugacy classes.

$k[G]^G = Z(k[G])$ (center consists of $f\in k[G]$ s.t.
$\delta_g * f = f * \delta_g \quad \forall g\in G$)

Proof.- $f = \sum_{g\in G} c_g \delta_g \in k[G]^G \iff \sum_{g\in G} c_g\delta_g = \sum_{g\in G} c_g\,\delta_{\sigma g\sigma^{-1}} \quad \forall \sigma\in G$

i.e. $c_g = c_{\sigma g \sigma^{-1}} \quad \forall\sigma\in G$.

On the other hand $\delta_\sigma * f = f * \delta_\sigma \iff \sum_{g\in G} c_g\,\delta_{\sigma g} = \sum_{g\in G} c_g\,\delta_{g\sigma}$

i.e., $$\sum_{h \in G} c_{\sigma^{-1} h}\, \delta_h = \sum_{h \in G} c_{h \sigma^{-1}}\, \delta_h$$

i.e. $c_h = c_{\sigma h \sigma^{-1}}$ $\forall\, h \in G$. Hence $k[G]^G = Z(k[G])$ $\square$

From the proof, it is clear that $k[G]^G$ has a basis given by

$$\left\{ \delta_C = \sum_{g \in C} \delta_g \ : \ \begin{array}{l} C \subset G \text{ is} \\ \text{a conjugacy class} \end{array} \right\}.$$ Hence, we have

$$\begin{array}{ccc}
k[G] & \xrightarrow[\psi]{\sim} & \bigoplus_{\lambda \in P(G)} \operatorname{End}_k(V_\lambda) \\
\cup & & \cup \\
Z(k[G]) & \xrightarrow{\sim} & \bigoplus_{\lambda \in P(G)} Z(\operatorname{End}_k(V_\lambda)) \\
\| & & \| \\
k[G]^G & \xrightarrow{\sim} & \bigoplus_{\lambda \in P(G)} k \cdot Id_{V_\lambda}
\end{array}$$

(basis labelled by conjugacy classes)

(basis labelled by the set of irred. f.d. repns $- P(G)$)

$\Rightarrow$ $|P(G)| =$ # of conjugacy classes in G.