Lecture 7

Let G be a finite group and k an alg. closed field s.t. char(k) does not divide $|G|$.

Recall: Notation: $\mathrm{Irr}_{fd}(G) = \{V_\lambda : \lambda \in P(G)\}$

$$k[G] \xrightarrow{\sim} \bigoplus_{\lambda \in P(G)} \mathrm{End}(V_\lambda)$$

$$\cup \qquad\qquad\qquad\qquad \cup$$

$$k[G]_{class} = Z(k[G]) \xrightarrow{\sim} \bigoplus_{\lambda \in P(G)} k \cdot \mathrm{Id}_{V_\lambda} = Z\Big(\bigoplus_{\lambda \in P(G)} \mathrm{End}(V_\lambda)\Big)$$

($k[G]_{class} = \{ f : G \to k \text{ s.t. } f(\sigma g \sigma^{-1}) = f(g)\ \forall \sigma, g \in G \}$
functions constant on conjugacy classes

has a basis $\{ \delta_C = \sum_{g \in C} \delta_g \mid C \subset G \text{ is a conjugacy class} \}$)

Hence, $|P(G)| = \#$ Conjugacy classes in G.

§1. Character of a finite-dim'l repn (Frobenius* 1896)

Let V be a f.d. repn. of a group G. Define $\chi_V : G \to k$ by

$$\chi_V(g) = \mathrm{Trace}\,(g \text{ acting on } V)$$
$$= \mathrm{Tr}(\rho_V(g)) \qquad (\rho_V : G \to GL(V) \text{ gp. hom.})$$

Note: since $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$, χ_V is a class function.

$$G \circlearrowright V \rightsquigarrow \chi_V \in k[G]_{class}.$$

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* Ferdinand Georg Frobenius (Oct. 26, 1849 - Aug. 3, 1917)

<u>Proposition.</u> (i) For every short exact sequence of G-repns (f.d.)

$$0 \to V_1 \to V \to V_2 \to 0, \quad \text{we have,}$$

$$\chi_V = \chi_{V_1} + \chi_{V_2}.$$

(ii) $$\chi_{V_1 \otimes V_2} = \chi_{V_1} \cdot \chi_{V_2}.$$

(iii) $$\chi_{V^*}(g) = \chi_V(g^{-1})$$

<u>Proof.</u> (i) The action of G on V is given by (NOT assuming semisimplicity, and viewing V as $V_1 \oplus V_2$ as k-vector space):

$$\rho_V(g) = \left[\begin{array}{c|c} \rho_{V_1}(g) & * \\ \hline 0 & \rho_{V_2}(g) \end{array}\right].$$ Hence $\mathrm{Tr}(\rho_V(g))$
$= \mathrm{Tr}(\rho_{V_1}(g)) + \mathrm{Tr}(\rho_{V_2}(g))$

(ii) follows from $\mathrm{Tr}(A \otimes B) = \mathrm{Tr}(A) \cdot \mathrm{Tr}(B)$ for $A \in M_{m \times m}(k)$
$B \in M_{n \times n}(k)$.

(if $v_1, \dots, v_m$ is a basis of k^m and $w_1, \dots, w_n$ a basis of k^n so that $A v_j = \sum_{i=1}^m a_{ij} v_i$ $\quad B w_s = \sum_{t=1}^n b_{ts} w_t$, then

$$(A \otimes B)(v_j \otimes w_s) = \sum_{i,t} a_{ij} b_{ts} \, v_i \otimes w_t$$

So, sum of diagonal entries of $A \otimes B = \sum_{j,s} a_{jj} b_{ss}$

$$= \Big(\sum_j a_{jj}\Big)\Big(\sum_s b_{ss}\Big) = \mathrm{Tr}(A)\,\mathrm{Tr}(B) \quad)$$

(iii) Let $v_1, \dots, v_n$ be a basis of V and $\eta_1, \dots, \eta_n$ be the dual basis of V^* (i.e. $\eta_j(v_i) = \delta_{ij}$)

Then $$\chi_{V^*}(g) = \sum_{i=1}^n \text{Coeff. of } \eta_i \text{ in } g \cdot \eta_i$$

$$\chi_{V^*}(g) = \sum_{i=1}^{n} (g\cdot\eta_i)(v_i) = \sum_{i=1}^{n} \eta_i\,(g^{-1}\cdot v_i)$$

$$= \sum_{i=1}^{n} \text{Coeff. of } v_i \text{ in } g^{-1}\cdot v_i = \operatorname{Tr}(\rho_V(g^{-1}))$$

$$= \chi_V(g^{-1}). \qquad \square$$

§2. Bilinear form on $k[G]_{class}$ and orthonormality of characters

Definition. – Let $\beta : k[G]_{class} \times k[G]_{class} \to k$ be defined by:

$$\beta(f_1, f_2) = \frac{1}{|G|} \sum_{\sigma\in G} f_1(\sigma)\, f_2(\sigma^{-1})$$

$$= \frac{1}{|G|} \sum_{g\in G} f_1(g^{-1})\, f_2(g) = \beta(f_2, f_1)$$

Then, β is a symmetric, bilinear form on $k[G]_{class}$.

Lemma. – Let $G \circlearrowright V, W$ be two f.d. repns. of G. Then,

$$\beta(\chi_V, \chi_W) = \dim \operatorname{Hom}_G(V, W)$$

Proof. – $\beta(\chi_V, \chi_W) = \dfrac{1}{|G|} \displaystyle\sum_{g\in G} \chi_V(g^{-1})\, \chi_W(g)$

$$= \frac{1}{|G|} \sum_{g\in G} \chi_{V^*}(g)\, \chi_W(g) \quad \text{(by (iii) of Prop. above)}$$

$$= \frac{1}{|G|} \sum_{g\in G} \chi_{V^*\otimes W}(g) \quad \text{(by (ii) of Prop. above)}$$

Hence $\beta(\chi_V, \chi_W)$ = Trace of $P = \frac{1}{|G|}\sum_{g\in G} g$ acting on $V^*\otimes W$.

Claim. Let $\rho: G \to GL(U)$ be a f.d. repn. Then $P = \frac{1}{|G|}\sum_{g\in G}\rho(g)$ is a G-intertwiner $P: U \to U^G = \{u\in U : g\cdot u = u \ \forall g\in G\}$ and $P|_{U^G} = \mathrm{Id}_{U^G}$. (Note: $U^G \subset U$ is a subrepn consisting of $\mathbb{1}$ (trivial repn) w/ mult = $\dim U^G$)

Hence $\mathrm{Tr}(P$ acting on $U) = \dim U^G$.

(Proof of the claim: for $u\in U$, $\sigma\in G$, we have

$$\sigma\cdot P(u) = \frac{1}{|G|}\sum_{g\in G}\rho(\sigma g)\,u = \frac{1}{|G|}\sum_{h\in G}\rho(h)u$$
$$= P(u). \quad \text{Hence, } P(u)\in U^G.$$

Moreover, if $u\in U^G \subset U$, the $P(u) = \frac{|G|}{|G|}u = u$. $\square$)

Hence,
$$\beta(\chi_V, \chi_W) = \dim (V^*\otimes W)^G$$
$$= \dim \mathrm{Hom}_k(V,W)^G \quad \left(\begin{array}{l}\text{via Tensor-Hom adjointness}\\ G \circlearrowright \mathrm{Hom}_k(V,W) \text{ via conjugation}\end{array}\right)$$
$$= \dim \mathrm{Hom}_G(V,W).$$

$\square$

§ 3. Corollary of Lemma §2 above. $\{\chi_\lambda = \chi_{V_\lambda} : \lambda\in P(G)\}$ is an orthonormal basis of $k[G]_{\text{class}}$ w.r.t. β (bilinear form)

In particular, β is non-degenerate.

Proof. $\beta(\chi_\lambda, \chi_\mu) = \dim \operatorname{Hom}_G (V_\lambda, V_\mu)$

$= \delta_{\lambda\mu}$ by Schur's Lemma. $\square$

§4. Further properties of characters.-

Theorem.- Let $f \in k[G]_{\text{class}}$. Define : $\rho_f : V \to V$

$\rho : G \circlearrowright V$ f.d. repn.

$$\rho_f = \sum_{\sigma \in G} f(\sigma^{-1})\rho(\sigma)$$

Assume V is irred. and let $\chi = \chi_V$

$n = \dim V$

Then $\rho_f = \lambda . \mathrm{Id}_V$ where $\lambda = \frac{|G|}{n} \beta(f, \chi)$

Proof.- $$\rho(\sigma) \circ \rho_f = \sum_{g \in G} f(g^{-1}) \rho(\sigma g) = \left(\sum_{g \in G} f(g^{-1}) \rho(\sigma g \sigma^{-1}) \right) \circ \rho(\sigma)$$

Now $f(g^{-1}) = f(\sigma g^{-1} \sigma^{-1}) \Rightarrow \rho(\sigma) \circ \rho_f = \rho_f \circ \rho(\sigma) \ \forall \sigma \in G$.

By Schur's lemma $\rho_f = \lambda . \mathrm{Id}_V$. Taking trace, we get

$$n \cdot \lambda = \sum_{\sigma \in G} f(\sigma^{-1}) \chi(\sigma) = |G| . \beta(f, \chi)$$

$\square$

Corollary.- Take $f = \delta_C$ where C is ~~a~~ the conjugacy class of $\sigma \in G$.

Write δ_C as a linear combination of $\{\chi_\lambda : \lambda \in P(G)\}$

$$\delta_C = \sum_{\lambda \in P(G)} c_\lambda \cdot \chi_\lambda \quad , \text{ where } c_\lambda \in k \text{ is}$$

given by
$$c_\lambda = \beta(\chi_\lambda, \delta_C) = \frac{1}{|G|} \sum_{g \in G} \chi_\lambda(g^{-1})\, \delta_C(g)$$

$$= \frac{1}{|G|} \sum_{g \in C} \chi_\lambda(g^{-1}) = \frac{|C|}{|G|} \chi_\lambda(\sigma^{-1}) \quad (\sigma \in C)$$

Hence,
$$\delta_C(g) = \sum_{\lambda \in P(G)} \frac{|C|}{|G|} \chi_\lambda(\sigma^{-1})\, \chi_\lambda(g) \quad \forall g \in G$$

For $g \in C$, we get
take $g = \sigma$

$$\boxed{\sum_{\lambda \in P(G)} \chi_\lambda(\sigma^{-1})\, \chi_\lambda(\sigma) = \frac{|C|}{|G|}}$$

For $g \notin C$ we get

$$\boxed{\sum_{\lambda \in P(G)} \chi_\lambda(\sigma^{-1})\, \chi_\lambda(g) = 0}$$

§5. Construction of projection operators.

Let $G \circlearrowright V$ be a f.d. repn ; $V = \bigoplus_{\lambda \in P(G)} V_\lambda^{\oplus d_\lambda(V)}$ $\quad (d_\lambda(V) \in \mathbb{Z}_{\geq 0})$

$$= \bigoplus_{\lambda \in P(G)} U_\lambda \quad \left(U_\lambda = V_\lambda^{\oplus d_\lambda(V)} \right)$$

<u>Thm.</u> $$P_\lambda := \frac{\dim(V_\lambda)}{|G|} \sum_{g \in G} \chi_\lambda(g^{-1})\, \rho_V(g) : V \to U_\lambda$$

(projection onto λ-th "isotypical" component)

Proof. It is enough to prove that P_λ acting on V_μ is 0 if $\mu \neq \lambda$ and Id_{V_λ} if $\mu = \lambda$. By Thm §4 above, P_λ acting on V_μ is a scalar matrix with eigenvalue

$$= \frac{|G|}{\dim V_\mu} \cdot \frac{\dim V_\lambda}{|G|} \beta(\chi_\lambda, \chi_\mu) = \delta_{\lambda\mu}.$$

□

§6. Inverse of $\psi : k[G] \longrightarrow \bigoplus_{\lambda \in P(G)} \mathrm{End}(V_\lambda)$ ("Fourier inversion formula")

Let ~~$\chi_\lambda \in \mathrm{End}$~~ $a_\lambda \in \mathrm{End}(V_\lambda)$ be given $(\forall \lambda \in P(G))$ and let $\alpha = \psi^{-1}((a_\lambda)) \in k[G]$. Then

$$\alpha = \sum_{g \in G} \alpha_g \delta_g \quad \text{where} \quad \boxed{\alpha_g = \frac{1}{|G|} \sum_{\lambda \in P(G)} (\dim V_\lambda) \, Tr(\rho_\lambda(g)^{-1} \cdot a_\lambda)}$$

Proof. - By linearity of ψ, and the fact that it is an iso, it is enough to verify the formula on $(a_\lambda) = \psi(\delta_\sigma)$ for some $\sigma \in G$. In this case $\alpha_g = \delta_{\sigma, g}$ and the right-hand side of the desired equation is

$$\frac{1}{|G|} \sum_{\lambda \in P(G)} \dim(V_\lambda) \, Tr(\rho_\lambda(g^{-1}\sigma))$$

$$= \frac{1}{|G|} \text{Trace of } g^{-1}\sigma \text{ on } \bigoplus_{\lambda \in P(G)} V_\lambda^{\oplus \dim V_\lambda} \simeq k[G]$$

$$= \begin{cases} \frac{|G|}{|G|} & \text{if } g^{-1}\sigma = e \\ 0 & \text{if } g^{-1}\sigma \neq e \end{cases} = \delta_{g,\sigma} \quad \text{as desired.}$$

□