Lecture 8

<u>Summary so far</u>: G denotes a finite group and k an algebraically closed field s.t. $|G| \neq 0$ in k.

$$\mathrm{Irr}_{fd}(G) = \{ \rho_\lambda : G \to GL(V_\lambda) \}_{\lambda \in P(G)} \qquad (|P(G)| = |\text{Conj. classes in } G|)$$

$$\begin{array}{ccc} \psi : k[G] & \xrightarrow{\sim} & \bigoplus_{\lambda \in P(G)} \mathrm{End}_k(V_\lambda) \\ \cup & & \cup \\ k[G]_{class} & \xrightarrow{\sim} & \bigoplus_{\lambda \in P(G)} k \cdot \mathrm{Id}_{V_\lambda} \end{array}$$

(Fundamental Thm. of repn. th. of finite gps.)

Structures on $k[G]$ = vector space of all k-valued fns on G:

- Convolution product: $(f_1 * f_2)(g) = \sum_{x \in G} f_1(x) f_2(x^{-1} g)$
- Bilinear (Symmetric) form: $\beta : k[G] \times k[G] \to k$ given by

$$\beta(f_1, f_2) = \frac{1}{|G|} (f_1 * f_2)(e) = \frac{1}{|G|} \sum_{g \in G} f_1(g^{-1}) f_2(g)$$

Recall: (Thm. §4 of Lecture 7)

For $f \in k[G]_{class}$ and $\lambda \in P(G)$, we have

$$\sum_{g \in G} f(g^{-1}) \rho_\lambda(g) = \frac{|G|}{\dim(V_\lambda)} \beta(f, \chi_\lambda) \cdot \mathrm{Id}_{V_\lambda}$$

- Characters and orthogonality relations.

for $\rho : G \to GL(V)$ a f.d. repn, $\chi_V : G \to k$ is defined as

$$\chi_V(g) = \mathrm{Tr}(\rho(g))$$

$$(\chi_V \in k[G]_{\text{class}}).$$

(i) $\beta(\chi_{V_\lambda}, \chi_{V_\mu}) = \delta_{\lambda\mu}$. In other words, for $\lambda, \mu \in P(G)$

$$\delta_{\lambda\mu} = \frac{1}{|G|} \sum_{g \in G} \chi_\lambda(g^{-1})\, \chi_\mu(g)$$

$$= \frac{1}{|G|} \sum_{C \in \mathrm{Conj}(G)} |C|\, \chi_\lambda(g_C^{-1})\, \chi_\mu(g_C)$$

[$\mathrm{Conj}(G)$ = set of conjugacy classes. $g_C \in C$ a representative]

(ii) $$\sum_{\lambda \in P(G)} \chi_\lambda(g^{-1})\, \chi_\lambda(h) = \begin{cases} 0 & \text{if } g \text{ and } h \text{ are not conjugate to each other}, \\ \dfrac{|G|}{|C|} & \text{if } g, h \in C \in \mathrm{Conj}(G). \end{cases}$$

§1. Orthogonality of matrix coefficients. -

Choose a basis $\{v_1^{(\lambda)}, \dots, v_{n_\lambda}^{(\lambda)}\}$ of V_λ $(n_\lambda = \dim V_\lambda)$

and let $\{\eta_1^{(\lambda)}, \dots, \eta_{n_\lambda}^{(\lambda)}\}$ be the dual basis of V_λ^*.

For each $i, j \in \{1, \ldots, n_\lambda\}$ define <u>the matrix coefficient</u>

$$c_{ij}^{(\lambda)} : G \to k \quad \text{by} \quad c_{ij}^{(\lambda)}(g) = \text{coefficient of } v_i^{(\lambda)} \text{ in } \rho_\lambda(g^{-1})(v_j^{(\lambda)}).$$

So, $g^{-1} \cdot v_j^{(\lambda)} = \sum_{i=1}^{n_\lambda} c_{ij}^{(\lambda)}(g) \cdot v_i^{(\lambda)}$. Alternately, using the dual basis $\{\eta_i^{(\lambda)}\}_{i=1}^{n_\lambda}$,

$$\boxed{c_{ij}^{(\lambda)}(g) = \eta_i^{(\lambda)}(g^{-1} \cdot v_j^{(\lambda)})}$$

<u>Theorem.</u> – For $\lambda, \mu \in P(G)$, $1 \leq i, j \leq n_\lambda$, $1 \leq s, t \leq n_\mu$; we have

$$\beta(c_{ij}^{(\lambda)}, c_{st}^{(\mu)}) = \begin{cases} 0 & \text{if } \lambda \neq \mu \\ \dfrac{\delta_{i,t}\,\delta_{j,s}}{n_\lambda} & \text{if } \lambda = \mu \end{cases}$$

<u>Proof.</u> – $\beta(c_{ij}^{(\lambda)}, c_{s,t}^{(\mu)}) = \frac{1}{|G|} \sum_{g \in G} \eta_i^{(\lambda)}(g \cdot v_j^{(\lambda)}) \cdot \eta_s^{(\mu)}(g^{-1} \cdot v_t^{(\mu)})$

$$= \frac{1}{|G|} \sum_{g \in G} \eta_i^{(\lambda)}(g \cdot v_j^{(\lambda)})\,(g \cdot \eta_s^{(\mu)})(v_t^{(\mu)})$$

Hence $\beta(c_{ij}^{(\lambda)}, c_{s,t}^{(\mu)}) = \left(\text{evaluation of } P = \frac{1}{|G|} \sum_{g \in G} g \otimes g \;(\text{on } V_\lambda \otimes V_\mu^*) \text{ at } v_j^{(\lambda)} \otimes \eta_s^{(\mu)}\right)$

paired with $\eta_i^{(\lambda)} \otimes v_t^{(\mu)}$

$$\begin{tikzcd}
V_\lambda \otimes V_\mu^* \arrow[rr, "P = \frac{1}{|G|}\sum_{g\in G} g\otimes g"] & & V_\lambda \otimes V_\mu^* \\
v_j^{(\lambda)} \otimes \eta_s^{(\mu)} \arrow[rr, mapsto] & & P\left(v_j^{(\lambda)} \otimes \eta_s^{(\mu)}\right) \arrow[d, "\eta_i^{(\lambda)} \otimes v_t^{(\mu)}"] \\
& & k \otimes k \arrow[r, "\text{mult.}"'] & k \ni \beta(c_{ij}^{(\lambda)}, c_{st}^{(\mu)})
\end{tikzcd}$$

Now $\text{Image}(P) = \left(V_\lambda \otimes V_\mu^*\right)^G$

$$= \text{Hom}_G(V_\mu, V_\lambda) = \begin{cases} \{0\} & \text{if } \mu \neq \lambda \\ k.\,\text{Id}_{V_\lambda} & \text{if } \mu = \lambda \end{cases} \quad \text{(Schur's lemma)}$$

$\Rightarrow \beta(c_{ij}^{(\lambda)}, c_{st}^{(\mu)}) = 0$ if $\lambda \neq \mu$. Assuming $\lambda = \mu$, we get

$$P\left(v_j^{(\lambda)} \otimes \eta_s^{(\lambda)}\right) = x_{j,s} \cdot \text{Id}_{V_\lambda} \qquad (x_{j,s} \in k)$$

$$= x_{j,s} \cdot \sum_{a=1}^{n_\lambda} v_a^{(\lambda)} \otimes \eta_a^{(\lambda)}$$

Apply the evaluation to get:
$(V_\lambda \otimes V_\lambda^* \to k)$

$$\frac{1}{|G|} \sum_{g\in G} (g\cdot\eta_s^{(\lambda)})(g\cdot v_j^{(\lambda)}) = n_\lambda \cdot x_{j,s}$$

i.e. $\eta_s^{(\lambda)}(v_j^{(\lambda)}) = n_\lambda \cdot x_{j,s} \Rightarrow x_{j,s} = \dfrac{\delta_{j,s}}{n_\lambda}$.

So, $$P\left(v_j^{(\lambda)} \otimes \eta_s^{(\lambda)}\right) = \frac{\delta_{j,s}}{n_\lambda} \cdot \sum_{a=1}^{n} v_a^{(\lambda)} \otimes \eta_a^{(\lambda)}$$

$$\Rightarrow \beta(c_{ij}^{(\lambda)}, c_{s,t}^{(\lambda)}) = \text{evaluate } \frac{\delta_{j,s}}{n_\lambda} \cdot \sum_{a=1}^{n_\lambda} v_a^{(\lambda)} \otimes \eta_a^{(\lambda)} \text{ on }$$

$$\eta_i^{(\lambda)} \otimes v_t^{(\lambda)}$$

$$= \frac{\delta_{j,s}\,\delta_{i,t}}{n_\lambda} \qquad \square$$

§2. Example of character table.- $G = S_3$.

Conjugacy class labelled by cycle type		# elts in a conj class	Representative	χ_1 (trivial repn.)
$1+1+1$	C_1	1	id	1
$2+1$	C_2	3	(12)	1
3	C_3	2	(123)	1

Similarly χ_{sgn} ($sgn : S_3 \to \{\pm 1\} \subset GL_1(k)$ 1-dim'l "sign repn")

	χ_{sgn}
C_1	1
C_2	-1
C_3	1

. If V is the remaining repn., $n = \dim V$ then $6 = 1^2 + 1^2 + n^2 \Rightarrow n = 2$.

	χ_V
C_1	2
C_2	x
C_3	y

We can compute x, y from $\beta(\chi_1, \chi_V) = 0 = \beta(\chi_{sgn}, \chi_V)$:

$$2 + 3x + 2y = 0 = 2 - 3x + 2y$$

$$\Rightarrow x = 0 \text{ and } y = -1.$$

Character table of S_3 :

Conj. classes \ Irr. repn		$\mathbb{1}$	sgn	V
$(1,1,1)$: C_1	$\boxed{1}$	1	1	2
$(2,1)$: C_2	$\boxed{3}$	1	-1	0
(3) : C_3	$\boxed{2}$	1	1	-1

elts.

Note : If $S_3 \circlearrowright \mathbb{C}^3$ is the standard repn. then $\mathrm{Tr}(\sigma|_{\mathbb{C}^3}) = \left|\{1,2,3\}^{\sigma}\right|$

$= \#$ of 1-cycles in σ

$$\chi_{\mathbb{C}^3} = \begin{matrix} 3 \\ 1 \\ 0 \end{matrix} = \chi_{\mathbb{1}} + \chi_V$$

i.e. $\mathbb{C}^3 \simeq \mathbb{1} \oplus V$ ← 2-dim'l complementary repn

$= \mathbb{C}$ - span of $\varepsilon_1 - \varepsilon_2$, $\varepsilon_2 - \varepsilon_3$

trivial repn.

$= \mathbb{C} \cdot (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$