Lecture 10

Representation theory of Symmetric groups. (over $\mathbb{C}$).

§1. Some basic facts and notations. Let $n \in \mathbb{Z}_{\geq 1}$ and S_n be the group of permutations of $\{1, \dots, n\}$.

(a) Cycles $(i_1, \dots, i_p) \in S_n$ denotes the permutation:

$$i_1 \mapsto i_2 \mapsto \dots \mapsto i_p \mapsto i_1 \quad \text{and} \quad j \mapsto j \text{ if } j \notin \{i_1, \dots, i_p\}$$

(b) Every permutation $w \in S_n$ can be written as a product of disjoint cycles (uniquely - up to reordering the cycles).

(c) For $w \in S_n$ and $(i_1, \dots, i_p) \in S_n$ a cycle, we have

$$w\,(i_1, \dots, i_p)\,w^{-1} = (w(i_1), \dots, w(i_p)).$$

Hence, cycle type of a permutation is invariant under conjugation.

$\uparrow$

tuple of positive integers = lengths of cycles appearing in a permutation

(e.g. cycle type of $(123)(46)(5)$ = $\underbrace{(3,2,1)}_{\text{partition of } 6}$)

(d) Let $\mathcal{P}(n)$ denote the set of partitions of n:

$$\mathcal{P}(n) = \left\{ \underline{\lambda} = (\lambda_1 \geq \dots \geq \lambda_\ell) \in \mathbb{Z}_{\geq 1}^{\ell} \ : \ \sum_{j=1}^{\ell} \lambda_j = n \right\}$$

(ℓ is called length of the partition λ)

Alternately, we can write a partition as $(\underbrace{1, \dots, 1}_{v_1\text{-times}}, \underbrace{2, \dots, 2}_{v_2\text{-times}}, \dots)$

So $\mathcal{P}(n) = \left\{ (v_1, v_2, \dots) \;\middle|\; \begin{array}{l} v_j \in \mathbb{Z}_{\geq 0} \ \forall j \\ \sum_{j=1}^{n} j v_j = n \end{array} \right\}$

(e) Conjugacy classes in S_n are labelled by partitions.

$$\mathrm{Conj}(S_n) \longleftrightarrow \mathcal{P}(n), \quad \text{where for } \underline{\lambda} = (\lambda_1 \geq \dots \geq \lambda_\ell > 0)$$

$\mathcal{C}(\underline{\lambda}) \subset S_n$ consist of all permutations of cycle type $\underline{\lambda}$.

(Cauchy):

$$\boxed{\begin{array}{l} |\mathcal{C}(\underline{\lambda})| = \dfrac{n!}{z(\underline{\lambda})} \quad \text{where} \\[2ex] z(\underline{\lambda}) = \prod_{i \geq 1} i^{v_i} \cdot v_i! \qquad (v_i = \#\{j \mid \lambda_j = i\}) \end{array}}$$

(left as an exercise)

§2. A family of S_n-repns. indexed by partitions.

For $\underline{\lambda} = (\lambda_1 \geq \dots \geq \lambda_\ell > 0) \in \mathcal{P}(n)$, consider the set

$$X(\underline{\lambda}) = \left\{ (I_1, \dots, I_\ell) \;\middle|\; \begin{array}{l} I_j \subset \{1, \dots, n\}; \ I_j \cap I_k = \phi \ \forall j \neq k \\ \{1, \dots, n\} = \bigsqcup_{j=1}^{\ell} I_j \ \underline{\text{and}} \\ |I_j| = \lambda_j \end{array} \right\}$$

(set of "partitions" of $\{1, \dots, n\}$ of fixed cardinality subsets)

e.g. $n = 4$, $\underline{\lambda} = (2, 2)$

$$X(2,2) = \left\{ \begin{array}{l} (\{1,2\}, \{3,4\}), \ (\{1,3\}, \{2,4\}), \ (\{1,4\}, \{2,3\}), \\ (\{3,4\}, \{1,2\}), (\{2,4\}, \{1,3\}), (\{2,3\}, \{1,4\}) \end{array} \right\}$$

As S_n acts naturally on $X(\underline{\lambda})$, we get a repn.

$$U(\underline{\lambda}) := \mathrm{Fun}(X(\underline{\lambda}), \mathbb{C})$$

Note : $\dim U(\underline{\lambda}) = |X(\underline{\lambda})| = \dfrac{n!}{\lambda_1! \cdots \lambda_\ell!}$

§3. <u>Character of $U(\underline{\lambda})$</u> Let $i_\lambda : S_n \to \mathbb{C}$ denote $\chi_{U(\underline{\lambda})}$ (character)

i.e., $i_\lambda(w)$ = Trace of w acting on $U(\underline{\lambda})$ $(w \in S_n)$

= # of w-fixed points in $X(\underline{\lambda})$ [see Problem 11]

$= |X(\underline{\lambda})^w|$

<u>Prop. (Frobenius, 1900)</u> Let $\underline{\mu} \in \mathcal{P}(n)$. Then

$i_\lambda(\mathcal{C}(\mu)) := i_\lambda(w_\mu)$ = coefficient of $x_1^{\lambda_1} \cdots x_\ell^{\lambda_\ell}$ in
$(w_\mu \in \mathcal{C}(\underline{\mu}))$ $\displaystyle\prod_{j \geq 1} (x_1^{\mu_j} + \cdots + x_N^{\mu_j})$

(here, $N \geq n$ is arbitrary).

<u>Proof</u>.- Let us pick $w_{\underline{\mu}} = (1\, 2 \cdots \mu_1)(\mu_1+1, \ldots, \mu_1+\mu_2) \cdots \in \mathcal{C}(\underline{\mu})$.

Let $J_1 = \{1, 2, \ldots, \mu_1\}$, $J_2 = \{\mu_1+1, \ldots, \mu_1+\mu_2\}$, ...

so that $\{1, \ldots, n\} = \bigsqcup_{i \geq 1} J_i$.

Then $X(\underline{\lambda})^{w_\mu} = \left\{ (I_1, \dots, I_\ell) \;\middle|\; \begin{array}{l} \{1,\dots,n\} = \bigsqcup_{j=1}^{\ell} I_j , \\ |I_j| = \lambda_j \text{, and each} \\ I_j \text{ is made up of blocks} \\ \text{from } J_1, \dots, J_r \end{array} \right\}$

(fixed points)

Easy check: $\left| X(\underline{\lambda})^{w_\mu} \right| = \text{coeff. of } x_1^{\lambda_1} \cdots x_\ell^{\lambda_\ell} \text{ in } \prod_{j=1}^{r} (x_1^{\mu_j} + \dots + x_N^{\mu_j})$ $\square$

e.g., $\lambda = (2,2) \in \mathcal{P}(4)$. (see page 2 above for the list of elements in $X(2,2)$)

$\mu = (2,1,1)$

$X(2,2)^{(12)} = \left\{ \{1,2\} \sqcup \{3,4\} , \{3,4\} \sqcup \{1,2\} \right\}$

(by direct inspection)

Those elements from $X(2,2)$, all of whose constituents can be written as union of some of $\{1,2\}$, $\{3\}$, $\{4\}$.

Coefficient of $x_1^2 x_2^2$ in $(x_1^2 + x_2^2 + x_3^2 + x_4^2)(x_1 + x_2 + x_3 + x_4)(x_1 + x_2 + x_3 + x_4)$
$= 2$.

§4. Example - Character table for $\{U_\lambda\}_{\lambda \in \mathcal{P}(n)}$: $(n=4)$

Conj(S_4) \ λ:		4	3+1	2+2	2+1+1	1+1+1+1
e	$\boxed{1}$	1	4	6	12	24
(12)	$\boxed{6}$	1	2	2	2	0
$(12)(34)$	$\boxed{3}$	1	0	2	0	0
(123)	$\boxed{8}$	1	1	0	0	0
(1234)	$\boxed{6}$	1	0	0	0	0

elts in given conj. class

$\boxed{\chi_{U_\lambda}(\mathcal{C}(\mu))}$

Observation: As we go from 1^{st} to 5^{th} column - we pick up exactly one more irred. repn. More precisely,

$U_{(4)} = \mathbb{1}$ trivial 1-dim'l repn - already irred. - denoted also by $V_{(4)}$

$$U_{(3,1)} = \mathbb{1} \oplus \underbrace{(\text{3-dim'l irred. repn})}_{\text{denote by } V_{(3,1)}}$$

$$U_{(2,2)} = \mathbb{1} \oplus V_{(3,1)} \oplus \underbrace{(\text{2-dim'l irred. repn})}_{\text{denote by } V_{(2,2)}}$$

$$U_{(2,1,1)} = \mathbb{1} \oplus V_{(3,1)}^{\oplus 2} \oplus V_{(2,2)} \oplus \underbrace{(\text{new 3-d. irr. rep})}_{\text{denote by } V_{(2,1,1)}}$$

$$U_{(1,1,1,1)} \ (= \mathbb{C}S_n) = \mathbb{1} \oplus V_{(3,1)}^{\oplus 3} \oplus V(2,2)^{\oplus 2} \oplus V_{(2,1,1)}^{\oplus 3} \oplus \underbrace{(\text{1-dim'l irred. repn})}_{\text{denote by } V_{(1,1,1,1)}}$$

$\rightsquigarrow$ actual character table (for irred. repns.)

	$\chi_{(4)}$	$\chi_{(3,1)}$	$\chi_{(2,2)}$	$\chi_{(2,1,1)}$	$\chi_{(1,1,1,1)}$
e	1	3	2	3	1
(12)	1	1	0	-1	-1
$(12)(34)$	1	-1	2	-1	1
(123)	1	0	-1	0	1
(1234)	1	-1	0	1	-1

§5. Ring of symmetric functions - various bases and statement of Frobenius' main theorem.

Let us keep number of variables $= N \geq n$ (otherwise immaterial).

$$\Lambda_{n,\mathbb{C}} = \mathbb{C}[x_1, \dots, x_N]^{S_N}_{\underset{\text{degree}}{n}}$$

= vector space of homogeneous degree n, symmetric polynomials in N variables.

- "Monomial" basis : given $\underline{\lambda} \in \mathcal{P}(n)$ define

$$m_{\underline{\lambda}} = \text{sum of all monomials in the } S_N\text{-orbit of } x_1^{\lambda_1} x_2^{\lambda_2} \cdots$$

e.g. $m_{(2,2)} = x_1^2 x_2^2 + x_1^2 x_3^2 + x_1^2 x_4^2 + x_2^2 x_3^2 + x_2^2 x_4^2 + x_3^2 x_4^2$

$(N = n = 4)$

- "power sum" basis : let $p_k = x_1^k + \dots + x_N^k$ and set

$$p_{\underline{\lambda}} = p_{\lambda_1} p_{\lambda_2} \cdots \quad \in \Lambda_{n,\mathbb{C}} \quad \forall\ \underline{\lambda} \in \mathcal{P}(n)$$

- "elementary symm. fns." let $e_k = \sum_{1 \leq i_1 < \dots < i_k \leq N} x_{i_1} \cdots x_{i_k}$ and set

$$e_{\underline{\lambda}} = e_{\lambda_1} e_{\lambda_2} \cdots \quad \in \Lambda_{n;\mathbb{C}} \quad \forall\ \underline{\lambda} \in \mathcal{P}(n)$$

- "complete symm. fns" let h_k = sum of all monomials (in $x_1, \dots, x_N$) of degree k.

$$h_{\underline{\lambda}} = h_{\lambda_1} h_{\lambda_2} \cdots \in \Lambda_{n;\mathbb{C}} \quad \forall \ \underline{\lambda} \in \mathcal{P}(n).$$

Frobenius characteristic map - Define :

$$(\mathbb{C} S_n)_{\text{class}} \xrightarrow{\text{ch}} \Lambda_{n;\mathbb{C}} \quad \text{by}$$

$$\text{ch}(\delta_{C(\underline{\mu})}) = \frac{p_{\underline{\mu}}}{z(\underline{\mu})} \quad \text{(power sum)}$$

Then : (i) $\text{ch}(U_{\underline{\lambda}}) = h_{\underline{\lambda}}$ (complete symm. fn.)

(ii) $\text{ch}(\underset{\uparrow}{V_\lambda}) = s_\lambda \leftarrow$ Schur polynomial (defined in the next lecture) see below

unique irred. appearing in U_λ

[$\displaystyle s_\lambda(x_1, \dots, x_n) := \frac{a(\lambda+\delta)}{a(\delta)}$ appeared in earlier works of Jacobi (1841) and Cauchy (1815).

$$\lambda+\delta = (\lambda_1+n-1, \lambda_2+n-2, \dots, \lambda_n+0) \quad \text{and}$$
$$\delta := (n-1, n-2, \dots, 0)$$

$$a(\alpha_1, \alpha_2, \dots, \alpha_n) := \det \begin{bmatrix} x_1^{\alpha_1} & x_2^{\alpha_1} & \dots & x_n^{\alpha_1} \\ x_1^{\alpha_2} & x_2^{\alpha_2} & \dots & x_n^{\alpha_2} \\ & \vdots & & \\ x_1^{\alpha_n} & x_2^{\alpha_n} & \dots & x_n^{\alpha_n} \end{bmatrix} \Bigg].$$