Lecture 13

Recall that we defined : (Lecture 12, §2, page 3)

$$S_\lambda(x_1, \dots, x_N) = \frac{a_{\lambda+\rho}(x_1, \dots, x_N)}{a_\rho(x_1, \dots, x_N)} \quad ; \text{ where,}$$

- $\lambda \vdash n$ and $N \geq n$.
- for $\underline{\alpha} \in \mathbb{N}^N$; $a_{\underline{\alpha}}(\underline{x}) = \det\left(x_i^{\alpha_j}\right)_{1 \leq i,j \leq N}$

$$= \sum_{w \in S_N} \varepsilon(w) \cdot x_1^{\alpha_{w(1)}} \cdots x_N^{\alpha_{w(N)}}$$

- $\rho = (N-1, \dots, 0) \in \mathbb{N}^N$ and $(\lambda+\rho)_j = \lambda_j + \rho_j = \lambda_j + N - j$

  $a_\rho(\underline{x}) = \prod_{i<j}(x_i - x_j)$ (note: $\lambda_j = 0$ if $j > \text{length}(\lambda)$)

The definition above is usually written in the form of "Weyl Character formula"

$$\boxed{S_\lambda(x_1, \dots, x_N) = \frac{\sum_{\sigma \in S_N} \varepsilon(\sigma)\, x^{\sigma(\lambda+\rho)}}{\prod_{i<j}(x_i - x_j)}}$$

Later we will see that the expression above is the trace of the diagonal element $\begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_N \end{pmatrix} \in GL_N(\mathbb{C})$ acting on an irreducible repn.: $GL_N(\mathbb{C}) \circlearrowright L_\lambda$ – labelled by partitions w/ length $\leq N$.

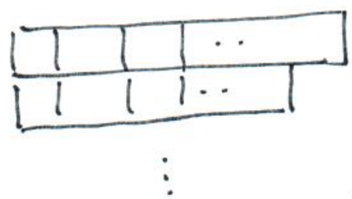
We also proved last time that

$$\boxed{S_\lambda = \det\left(h_{\lambda_i - i + j}\right)_{1 \le i,j \le l = \text{length}(\lambda)}}$$

(Lecture 12, §3, page 5) (Jacobi-Trudi identity)

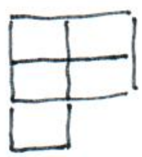
The following result implies positivity of coefficients of S_λ - expressed in terms of monomial symmetric polynomials.

§1. <u>Young tableaux</u>. - It is convenient to draw a partition as a "Young diagram" $Y(\lambda)$:



λ_1 boxes
λ_2 boxes

e.g. $Y(2,2,1)$ =



. λ is often called the "shape of $Y(\lambda)$".

A <u>semi-standard Young tableaux</u> of shape λ is an assignment of positive integer to each box of $Y(\lambda)$ s.t. the numbers <u>weakly increase along rows</u>, and <u>strictly increase along columns</u>.

e.g.

$$\begin{array}{c|c|c|}
 & \multicolumn{2}{c}{\le} \\
\cline{2-3}
 & 1 & 1 \\
\cline{2-3}
\wedge & 2 & 3 \\
\cline{2-3}
 & 7 & \multicolumn{1}{c}{} \\
\cline{2-2}
\end{array}$$

a semi-standard Young tableaux of shape (2,2,1)

$$\begin{array}{|c|c|c|}
\hline
1 & 1 & 2 \\
\hline
1 & \multicolumn{2}{c}{} \\
\cline{1-1}
\end{array}$$

<u>NOT</u> semi-standard

$SSYT_N(\lambda)$ = set of all semi standard Young tableau of shape λ, filled with numbers from $\{1, \dots, N\}$.

For $t \in SSYT_N(\lambda)$, let type of t be the array of non-negative integers $\quad v_i(t) = \#$ of i's in $t \quad (1 \leq i \leq N)$

$v(t) \in \mathbb{N}^N$.

Theorem. - (Young's[*] rule). -

$$\boxed{S_\lambda(x_1, \dots, x_N) = \sum_{t \in SSYT_N(\lambda)} \underline{x}^{v(t)}}$$

Hence, coefficient of m_μ in S_λ is the number of semi-standard Young tableau of shape λ and type μ. This number is called Kostka number:

$$K_{\lambda\mu} = \#\{t \in SSYT_N(\lambda) : v(t) = \mu\} \quad (N \geq n)$$

$$S_\lambda = \sum_{\mu \vdash n} K_{\lambda\mu} m_\mu .$$

The proof of this theorem given in §3 below is based on a combinatorial lemma due to Lindstrøm and Gessel-Viennot.

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* Alfred Young (April 16, 1873 - December 15, 1940)

§2. Determinant of weighted path matrices.

Assume (V, E) is a directed, acyclic graph; and $\{x_e : e \in E\}$ is a set of (commuting) variables.

Let $A = \{a_1, \dots, a_\ell\}$ and $B = \{b_1, \dots, b_\ell\}$ be two subsets of the set of vertices. Define the weighted path matrix $\mathcal{P}(A|B)$ to be $\ell \times \ell$ matrix with entries from $\mathbb{Z}[x_e : e \in E]$, given by:

$$P_{ij} = \sum_{\substack{\text{paths:} \\ p:\, i \xrightarrow{e_1} \cdots \xrightarrow{e_r} j}} wt(p) \quad ; \quad wt(p) = \prod_{e \in p} x_e$$

(product of weights x_e of edges appearing in the path p)

Lemma.- (Lindstrøm; Gessel-Viennot)

$$\det(\mathcal{P}(A|B)) = \sum_{(\sigma;\, \underline{\pi})} \varepsilon(\sigma)\, wt(\pi_1) \cdots wt(\pi_\ell)$$

sum is over all $\sigma \in S_\ell$ and $\underline{\pi} = (\pi_1, \dots, \pi_\ell)$ where

π_i is a path from a_i to $b_{\sigma(i)}$ s.t.

π_i and π_j do not share any vertex ($\forall\, i \neq j$)

(i.e. $\pi_1, \dots, \pi_\ell$ is a tuple of non-intersecting paths $a_1 \xrightarrow{\pi_1} b_{\sigma(1)}; \dots, a_\ell \xrightarrow{\pi_\ell} b_{\sigma(\ell)}$).

Proof.- By definition of determinant, we have,

$$\det \mathcal{P}(A \mid B) = \sum_{(\sigma,\, \underline{\pi} = (\pi_1, \dots, \pi_\ell))} \varepsilon(\sigma)\, wt(\pi_1) \dots wt(\pi_\ell) ,$$

where the sum is over all $\sigma \in S_\ell$ and $\pi = (\pi_1, \dots, \pi_\ell)$ is a tuple of paths $\pi_j : a_j \to b_{\sigma(j)}$. Let us call this set $\mathcal{E}$. Write

$$\mathcal{E} = \mathcal{I} \sqcup \mathcal{N} \text{ where } \mathcal{N} = \left\{ (\sigma, \underline{\pi}) \;\middle|\; \begin{array}{c} \pi_1, \dots, \pi_\ell \text{ are} \\ \text{non-intersecting} \end{array} \right\}$$

So, $$\det \mathcal{P}(A \mid B) = \sum_{(\sigma, \underline{\pi}) \in \mathcal{I}} \varepsilon(\sigma)\, wt(\underline{\pi}) + \sum_{(\sigma, \underline{\pi}) \in \mathcal{N}} \varepsilon(\sigma)\, wt(\underline{\pi}).$$

We <u>claim</u> that <u>there is an involution</u> $\phi : \mathcal{I} \to \mathcal{I}$ $(\phi \circ \phi = Id_{\mathcal{I}})$

s.t. $\phi(\sigma, \underline{\pi}) = (\sigma', \underline{\pi}') \Rightarrow \begin{array}{l} \varepsilon(\sigma') = -\varepsilon(\sigma) \\ \& \ wt(\underline{\pi}) = wt(\underline{\pi}) \end{array}$.

Hence $\displaystyle\sum_{(\sigma, \underline{\pi}) \in \mathcal{I}} \varepsilon(\sigma)\, wt(\underline{\pi}) = 0$ and the lemma follows.

<u>Proof of the claim</u> - i.e., construction of $\phi : \mathcal{I} \to \mathcal{I}$:

Let $(\sigma, \underline{\pi}) \in \mathcal{I}$. Pick $i_0 \in \{1, \dots, \ell\}$ smallest such that π_{i_0} shares a vertex with some π_j. Let $v \in V$ be the first vertex along π_{i_0} which also lies on other paths. Let $j_0 \in \{1, \dots, \ell\}$ be the smallest such that π_{i_0} and π_{j_0} meet at vertex v.

Define $(\pi_1', \dots, \pi_\ell')$ by setting $\pi_k' = \pi_k$ if $k \neq i_0, j_0$.

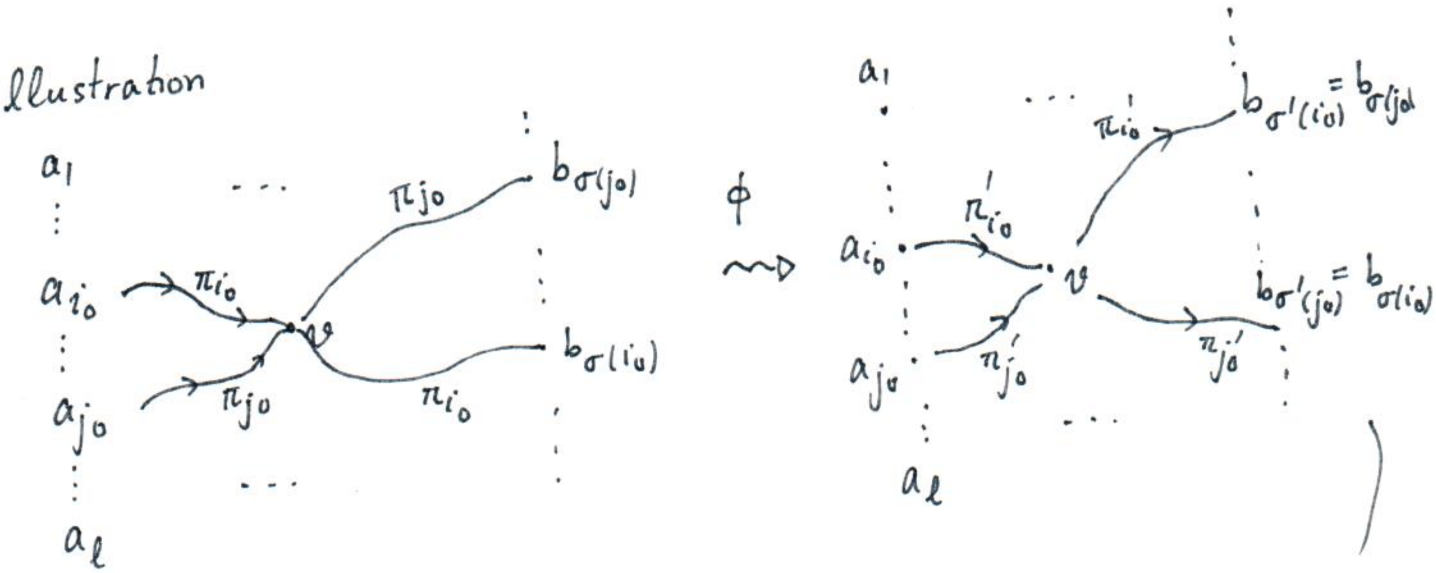
π_{i_0}' : follow π_{i_0} until vertex v and then follow π_{j_0}.

π_{j_0}' : " π_{j_0} " " " " " " π_{i_0}.

The resulting permutation: $\sigma' = \sigma \circ (i_0\, j_0)$, hence $\varepsilon(\sigma') = -\varepsilon(\sigma)$.

Thus $\phi(\sigma, \underline{\pi}) = (\sigma', \underline{\pi}')$ satisfies the required conditions. (Check: $\phi(\phi(\sigma, \underline{\pi})) = (\sigma, \underline{\pi})$.) $\square$

(Illustration



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§3. <u>Proof of Young's rule</u>. – Consider the following graph:

$$V = \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}. \quad \text{Edge set} = \left\{ \begin{array}{c} (i,j) \xrightarrow{x_j \text{ (weight)}} (i+1, j) \\[2ex] (i, j+1) \\ \uparrow \text{weight } 1 \\ (i,j) \end{array} \right\}_{(i,j) \in V}$$

Then weighted sum of all paths $(i,1) \to (i+\ell, N)$

$= $ sum of all monomials in $\{x_1, \dots, x_N\}$ of degree ℓ

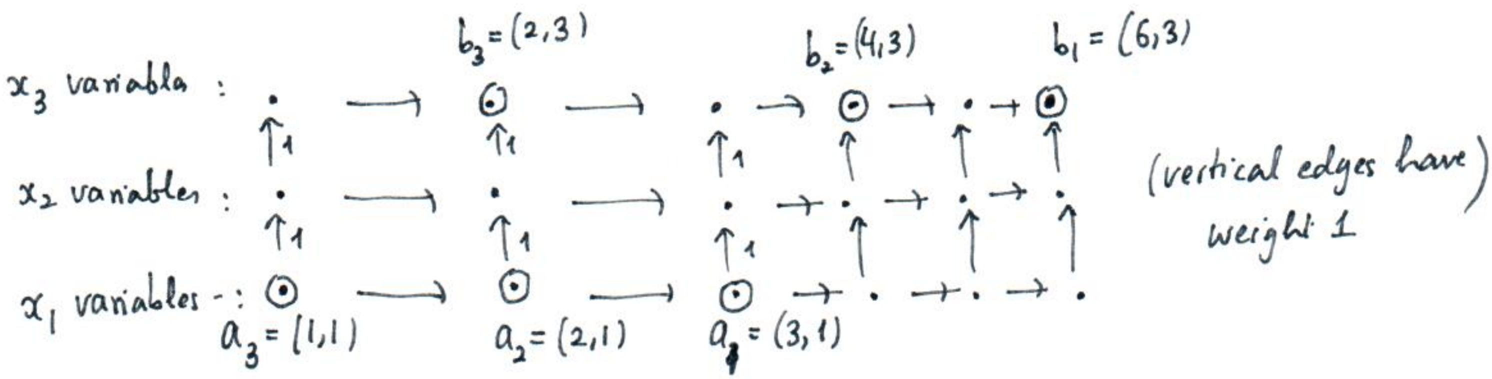
$= h_\ell(x_1, \dots, x_N)$. Given $\lambda \vdash n$ define:

Source set $A = \{ a_1 = (\ell, 1),\ \cancel{b_1 \neq (\ell,}\ , a_2 = (\ell-1, 1), \dots, a_\ell = (1,1)\}$

Target set $B = \{ b_1 = (\ell + \lambda_1, N),\ b_2 = (\ell - 1 + \lambda_2, N), \dots, b_\ell = (1 + \lambda_\ell, N)\}$

Then $P(A \mid B) = \left(h_{\lambda_i - i + j} \right)_{1 \le i,j \le \ell}$

e.g. $\lambda = (3,2,1)$ $(N = 3)$



$a_1 \dashrightarrow b_1 : h_3$ $a_2 \dashrightarrow b_1 : h_4$ $a_3 \dashrightarrow b_1 : h_5$

$a_1 \dashrightarrow b_2 : h_1$ $a_2 \dashrightarrow b_2 : h_2$ $a_3 \dashrightarrow b_2 : h_3$

$a_1 \dashrightarrow b_3 : h_{-1} = 0$ $a_2 \dashrightarrow b_3 : h_0 = 1$ $a_3 \dashrightarrow b_3 : h_1$

$$P(A \mid B) = \begin{bmatrix} h_3 & h_1 & 0 \\ h_4 & h_2 & 1 \\ h_5 & h_3 & h_1 \end{bmatrix} \rightsquigarrow \det = S_{(3,2,1)}.$$

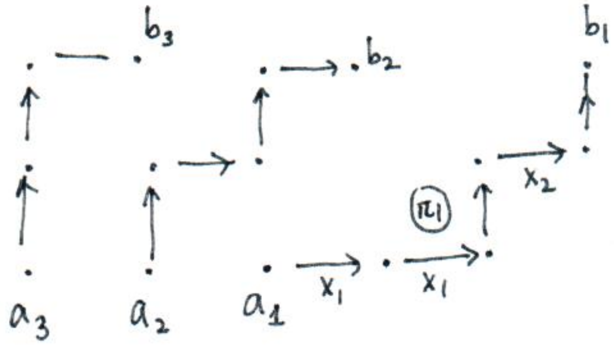
Finally, we claim that there is a bijection:

Set of ℓ-tuples of non-intersecting paths $\pi_1 : a_1 \dashrightarrow b_1$, $\vdots$, $\pi_\ell : a_\ell \dashrightarrow b_\ell$ $\longleftrightarrow$ $SSYT_N(\lambda)$

$\underline{\pi} = (\pi_1, \dots, \pi_\ell) \longrightarrow$ horizontal steps from π_i in the i-th row of $Y(\lambda)$. : $t(\underline{\pi})$

(Ex. - Verify that the map written above is a bijection.)

e.g.



horizontal steps from π_1.

1	1	2
2	3	
3		

$\lambda = (3, 2, 1)$

$N = 3$

and $wt(\pi_1) \cdots wt(\pi_\ell) = \underline{x}^{\nu(t(\underline{\pi}))}$. $\square$

The theorem from §1 follows.