Lecture 14.

Recall: S_n = symmetric group on n letters.

- $\mathrm{Conj}(S_n) \leftrightarrow \mathcal{P}(n)$ = set of partitions of n $\leftrightarrow \mathrm{Irr}_{fd}(S_n)$ (over $\mathbb{C}$, say)
- $(\mathbb{C}S_n)_{\text{class}} = \{ f : S_n \to \mathbb{C} \mid f(\sigma w \sigma^{-1}) = f(w)\ \forall \sigma, w \in S_n \}$.
- $\forall$ f.d. repn $S_n \overset{\alpha}{\circlearrowright} V$; $\chi_V : S_n \to \mathbb{C}$ "character of V"

  $\chi_V(w)$ = Trace of $\alpha(w) \in GL(V)$

  (Note, $\chi_V \in (\mathbb{C}S_n)_{\text{class}}$)

$R(S_n) :=$ $\mathbb{Z}$ - span of $\{ \chi_V : V \in \mathrm{Rep}_{fd}(S_n ; \mathbb{C}) \}$

- Non-degenerate, symmetric, bilinear form $(\cdot,\cdot) : \mathbb{C}S_n \times \mathbb{C}S_n \to \mathbb{C}$;

restricted to $(\mathbb{C}S_n)_{\text{class}}$ becomes:

$$(f, g) = \sum_{\lambda \vdash n} \frac{f(\lambda)\, g(\lambda)}{z(\lambda)}$$

[$f(\lambda)$ = value of f at any element from conjugacy class $C(\lambda)$]

$$z(\lambda) = \prod_{j \geq 1} j^{\ell_j} \cdot \ell_j! \quad \left(\ell_j = \#\{ k : \lambda_k = j \} \right)$$

$= \left| \text{Centralizer of any element } w_\lambda \in C(\lambda) \right|$

Recall: $(\chi_V, \chi_W) = \dim \mathrm{Hom}_{S_n}(V, W)$

$= \sum_{\lambda \vdash n} d_\lambda(V) \cdot d_\lambda(W)$; where $V = \bigoplus_{\lambda \vdash n} V_\lambda^{\oplus d_\lambda(V)}$

$W = \bigoplus_{\lambda \vdash n} V_\lambda^{\oplus d_\lambda(W)}$

$\{ V_\lambda \}_{\lambda \in \mathcal{P}(n)} = \mathrm{Irr}_{fd}(S_n)$.

Thus, $\{\chi_\lambda = \chi_{V_\lambda}\}_{\lambda \vdash n}$ form an orthonormal basis of $(\mathbb{C}S_n)_{class}$ and $\mathbb{Z}$-basis of $R(S_n) \subset (\mathbb{C}S_n)_{class}$.

- <u>Frobenius characteristic map</u> $\quad Ch : (\mathbb{C}S_n)_{class} \to \Lambda_{n;\mathbb{C}}$

$$Ch(\delta_{c(\lambda)}) = \frac{p_\lambda}{z(\lambda)}$$

encodes class function as "generating series"

$$Ch(f) = \sum_{\lambda \vdash n} f(\lambda) \cdot \frac{p_\lambda}{z(\lambda)}$$

[Recall: $\Lambda_n = \mathbb{Z}[x_1, \ldots, x_N]^{S_N}$ degree n.

$p_r = \sum_{j=1}^{N} x_j^r \quad (N \geq n) \quad (r \geq 1)$

$p_\lambda = p_{\lambda_1} \cdots p_{\lambda_\ell}$]

We have proved the following:

(a) Ch is an isometry – i.e., preserves non-deg, symmetric bilinear forms (when $(\cdot,\cdot) : \Lambda_{n;\mathbb{C}} \times \Lambda_{n;\mathbb{C}} \to \mathbb{C}$.)

$$(p_\lambda, p_\mu) = \delta_{\lambda\mu} z(\lambda)$$

(b) Canonical tensor of the bilinear form on $\Lambda_{n;\mathbb{C}}$ is given by:

$$\Omega(\underline{x}, \underline{y}) = \prod_{i,j} \frac{1}{1 - x_i y_j} \quad \text{(degree } n \text{ component)}$$

Schur polynomials $\{s_\lambda(\underline{x})\}_{\lambda \vdash n}$ form an orthonormal basis of $\Lambda_{n;\mathbb{C}}$
[Thm. §2 of Lecture 12]

$$s_\lambda(\underline{x}) = \det\left(h_{\lambda_i - i + j}\right)_{1 \leq i,j \leq \ell} = \sum_{\mu \vdash n} K_{\lambda\mu} m_\mu$$

(Lecture 12, §3, page 5) $\qquad$ (Young's rule Lecture 13, page 3)

recall $h_r = \sum_{\lambda \vdash r} m_\lambda$ (sum of all monomials of degree r)

$$m_\mu = \sum_{\alpha \in S_N \cdot \mu} \underline{x}^{\alpha} = x_1^{\mu_1} x_2^{\mu_2} \cdots x_k^{\mu_k} + \text{its symmetrization over } N \text{ variables.}$$

§1. $K_{\lambda\mu} = \#\{$ Semi-standard Young tableaux of shape λ and type μ – i.e., filled with μ_1 1's, μ_2 2's, $\vdots$, μ_K K's $\}$

Note – if $K_{\lambda\mu} \neq 0$ then $\mu_1 \leq \lambda_1$ and $\mu_1 + \mu_2 \leq \lambda_1 + \lambda_2$ ...

since # of 1's + ... + # r's in a SSYT of shape λ & type μ $= \mu_1 + \dots + \mu_r$ entries have to go in the first r rows $(\forall r)$

$\Rightarrow \mu_1 + \dots + \mu_r \leq \lambda_1 + \dots + \lambda_r$ $(\forall r)$. This is <u>a partial order</u>, (called dominance order) denoted here by $\underset{D}{\leq}$.

(D for dominance)

Thus, $s_\lambda = \sum_{\mu \underset{D}{\leq} \lambda} K_{\lambda\mu} m_\mu$ is a triangular transformation.

Moreover $K_{\lambda\lambda} = 1$ (check). Hence we get

<u>Cor of Young's rule</u> – $\{s_\lambda\}$ form a $\mathbb{Z}$-basis of Λ_n.

§2. We will now show that Frobenius' characteristic map preserves the $\mathbb{Z}$-forms.

$$\begin{array}{ccc} (\mathbb{C}S_n)_{\text{class}} & \longrightarrow & \Lambda_{n;\mathbb{C}} \\ \cup & & \cup \\ R(S_n) & \dashrightarrow & \Lambda_n \end{array}$$

i.e. To prove: $Ch(R(S_n)) = \Lambda_n$.

Recall – we constructed repns

$$\{U_\lambda = Fun(X_\lambda; \mathbb{C})\}_{\lambda \vdash P(n)}$$

$$X_\lambda = \{ I_1 \sqcup \dots \sqcup I_\ell = \{1, \dots, n\} : |I_j| = \lambda_j \ \forall j \}$$

$$\left\{ i_\lambda = \chi_{U_\lambda} \in R(S_n) \right\}_{\lambda \vdash n}$$ (we don't know if it is a $\mathbb{Z}$-basis yet).

Recall: $$i_\lambda(\mu) = \#\left\{ (I_1, \dots, I_\ell) \in X_\lambda : \begin{array}{l} \text{each } I_j \text{ can be written as} \\ \text{a union of } J_r\text{'s} \end{array} \right\}$$

$$J_1 = \{1, \dots, \mu_1\}; \; J_2 = \{\mu_1 + 1, \dots, \mu_1 + \mu_2\}; \dots$$

$= $ coefficient of m_λ in p_μ.

§3. Theorem. – (i) $\mathrm{Ch}(i_\lambda) = h_\lambda$.

(ii) $\Omega(\underline{x}, \underline{y}) = \sum_\lambda h_\lambda(\underline{x})\, m_\lambda(\underline{y})$. That is,

$\{h_\lambda\}_{\lambda \vdash n}$ form a $\mathbb{Z}$-basis of Λ_n – dual to $\{m_\lambda\}_{\lambda \vdash n}$:

$$(h_\lambda, m_\mu) = \delta_{\lambda\mu}.$$

Proof. – $$\mathrm{Ch}(i_\lambda) = \sum_{\mu \vdash n} i_\lambda(\mu) \cdot \frac{p_\mu}{z(\mu)}$$ (by definition).

$$= \sum_{\mu \vdash n} \frac{i_\lambda(\mu)}{z(\mu)} \sum_{\gamma \vdash n} i_\gamma(\mu)\, m_\gamma$$ $\left(\text{since } p_\mu = \sum_{\gamma \vdash n} m_\gamma \cdot i_\gamma(\mu)\right)$

$$= \sum_{\gamma \vdash n} m_\gamma \cdot \sum_{\mu \vdash n} \frac{i_\lambda(\mu)\, i_\gamma(\mu)}{z(\mu)}$$

$$= \sum_{\gamma \vdash n} (i_\lambda, i_\gamma)\, m_\gamma$$

The proof of the theorem follows from the following enumerative result.

§4. Matrices with fixed row & column sums. – Let $\underline{\alpha}, \underline{\beta} \in \mathbb{Z}_{\geq 0}^{\ell}$.

$$\mathcal{M}(\underline{\alpha};\underline{\beta}) = \left\{ A = (a_{ij}) \in M_{\ell\times\ell}(\mathbb{Z}_{\geq 0}) : \begin{array}{l} \sum_j a_{ij} = \alpha_i \quad (\forall\, 1\leq i\leq \ell) \\ \sum_i a_{ij} = \beta_j \quad (\forall\, 1\leq j\leq \ell) \end{array} \right\}$$

(of necessity; $\sum_i \alpha_i = \sum_j \beta_j$; otherwise $\mathcal{M}(\underline{\alpha};\underline{\beta}) = \phi$.)

Lemma. Let $\lambda, \mu \vdash n$ and $M_{\lambda\mu} = |\mathcal{M}_{\cancel{x}}(\lambda;\mu)|$. Then

(a) $(1_\lambda, 1_\mu) = M_{\lambda\mu}$

(b) $h_\lambda = \sum_{\mu\vdash n} M_{\lambda\mu}\, m_\mu$

(c) degree n part of $\prod_{i,j} \frac{1}{1-x_i y_j} = \sum_{\lambda,\mu\vdash n} M_{\lambda\mu}\, m_\lambda(x)\, m_\mu(y)$
$$= \sum_{\lambda\vdash n} m_\lambda(x)\, h_\lambda(y)$$

Proof. – (a)

$$(1_\lambda, 1_\mu) = \dim \mathrm{Hom}_{S_n}(U_\lambda, U_\mu)$$

$\neq$; ~~dim Hom~~ ~~λ~~ Recall: $U_\lambda = \mathrm{Fun}(X_\lambda;\mathbb{C})$

$\mathrm{Hom}_{\mathbb{C}}(U_\lambda, U_\mu) \simeq \mathrm{Fun}(X_\lambda \times X_\mu;\mathbb{C})$ as S_n-repns (check this!)

$E_{ba} \longmapsto \delta_{(a,b)}$ ← (delta fn. at $(a,b)\in X_\lambda\times X_\mu$)

$(a\in X_\lambda,\ b\in X_\mu)$

(E_{ba} maps the basis vector δ_c to $\boxed{\delta_{a,c}}\cdot\delta_b$ ← Kronecker δ-fn.)

U_λ has basis $\{\delta_a : a\in X_\lambda\}$

U_μ —— $\{\delta_b : b\in X_\mu\}$

$$\Rightarrow \quad (i_\lambda, i_\mu) = \dim \operatorname{Fun}(X_\lambda \times X_\mu ; \mathbb{C})^{S_n}$$
$$= |\, S_n\text{-orbits in } X_\lambda \times X_\mu \,|$$

Exercise – The map
$$X_\lambda \times X_\mu \longrightarrow \mathcal{M}_{\lambda\mu}(\lambda;\mu)$$
$$((I_1,\dots,I_\ell);(J_1,\dots,J_\ell)) \mapsto (|I_i \cap J_j|)_{1\le i,j\le \ell}$$
$$(\ell \ge \operatorname{length}(\lambda), \operatorname{length}(\mu))$$

sets up a bijection b/w S_n-orbits in $X_\lambda \times X_\mu$ and $\mathcal{M}(\lambda;\mu)$.
This proves (a).

(b) Coefficient of m_μ in h_λ = coefficient of $x_1^{\mu_1} \cdots x_\ell^{\mu_\ell}$ in
$$\Big(\sum_{|\alpha^{(1)}|=\lambda_1} \underline{x}^{\alpha^{(1)}}\Big)\Big(\sum_{|\alpha^{(2)}|=\lambda_2} \underline{x}^{\alpha^{(2)}}\Big)\cdots\Big(\sum_{|\alpha^{(\ell)}|=\lambda_\ell} \underline{x}^{\alpha^{(\ell)}}\Big)$$
$$= \sum (x_1^{a_{11}} x_2^{a_{12}} \cdots)(x_1^{a_{21}} x_2^{a_{22}} \cdots)\cdots$$
(sum is over all non-negative integer tuples s.t. $\sum_j a_{ij} = \lambda_i$)

Hence coeff. of m_μ $= |\mathcal{M}(\lambda;\mu)| = M_{\lambda\mu}$.

(c)
$$\prod_{i,j} \frac{1}{1-x_i y_j} = \prod_{i,j} (1 + x_i y_j + x_i^2 y_j^2 + \cdots)$$
$$= \sum (x_i y_j)^{a_{ij}} \qquad \text{(sum over all } A=(a_{ij}) \text{ matrix with } \mathbb{Z}_{\ge 0}\text{-entries)}$$

Coefficient of $m_\lambda(x)\, m_\mu(y)$ comes from $A=(a_{ij})$ s.t. $\sum_j a_{ij} = \lambda_i$, $\sum_i a_{ij} = \mu_j$
$= M_{\lambda,\mu}$.

$\square$