Lecture 15

§1. Induced representations.-

Let G be a finite group and $H < G$ a subgroup. Note that any representation of G can be considered as a representation of H, via restriction. [Here, repns. are considered over $\mathbb{C}$].

$$\alpha : G \to GL(V) \rightsquigarrow \alpha|_H : H \to GL(V)$$

Thus, defining a functor $\mathrm{Res}_H^G : \mathrm{Rep}(G) \to \mathrm{Rep}(H)$.

Now, let $\rho : H \to GL(W)$ be a repn. of H

<u>Definition</u>.- <u>Induced representation</u> $V = \mathrm{Ind}_H^G(W)$ is defined as:

$$V = \{ f : G \to W \text{ s.t. } f(gh) = \rho(h)^{-1} \cdot f(g) \ \forall \, {}^{g \in G}_{h \in H} \}$$

(as vector space)

$G \circlearrowright V$ is given by:

$$(g \cdot f)(\sigma) = f(g^{-1}\sigma) \quad \forall f \in V; \ g, \sigma \in G.$$

[Check: $g \cdot f \in V$; i.e. $(g \cdot f)(\sigma h) = \rho(h)^{-1} \cdot ((g \cdot f)(\sigma))$

$$(g \cdot f)(\sigma h) = f(g^{-1}\sigma h) = \rho(h)^{-1} \cdot f(g^{-1}\sigma) \ \checkmark \ ]$$

If $\varphi : W_1 \to W_2$ is an H-intertwiner, we get

$$\left.\begin{aligned} \mathrm{Ind}_H^G(\varphi) : \mathrm{Ind}_H^G(W_1) &\to \mathrm{Ind}_H^G(W_2) \\ f &\longmapsto \varphi \circ f \end{aligned}\right\} \text{ G-intertwiner.}$$

Ex: Check that Ind_H^G defined above gives <u>a functor</u>

$$\mathrm{Rep}(H) \to \mathrm{Rep}(G).$$

§2. Examples and remarks.-

(i) Partition repns.- $G = S_n$ (symmetric group) ; $\lambda \vdash n$.

$$X_\lambda = \{(I_1, \dots, I_\ell) \mid \{1,\dots,n\} = \bigsqcup_{j=1}^{\ell} I_j \ ; \ |I_j| = \lambda_j \ \forall j\}$$

$$\longleftrightarrow \quad S_n/S_\lambda \quad \text{where } S_\lambda = S_{\lambda_1} \times \dots \times S_{\lambda_\ell} < S_n.$$

The partition repn U_λ from Lecture 10 is $\mathrm{Ind}_{S_\lambda}^{S_n}$ (Trivial).

(ii) The process of taking induced repns. changes the underlying vector space. If $\dim W < \infty$, then,

$$\dim\left(\mathrm{Ind}_H^G W\right) = (\dim W)\cdot |G/H|.$$

(iii) Let $G = D_n$ (dihedral group $|D_n| = 2n$).

$$H = \langle r \rangle \cong \mathbb{Z}/n\mathbb{Z} \quad < \quad G = D_n = \langle r, s \mid s^2 = r^n = (sr)^2 = e \rangle$$

Take $W = \mathbb{C}$ with $\rho_z : H \to GL(\mathbb{C}) = \mathbb{C}^\times$ $\quad (z \in \mathbb{C}^\times \text{ s.t. } z^n = 1)$

$$\rho_z(r) = z$$

$$V = \mathrm{Ind}_H^G(W) = \{f : G \to \mathbb{C} \text{ s.t. } f(wr) = z^{-1} f(w) \ \forall w \in G\}$$

As $D_n = \left\{ \begin{matrix} e, r, r^2, \dots, r^{n-1} \\ s, sr, \dots, sr^{n-1} \end{matrix} \right\}$, V is 2-dim'l

$$V \simeq \mathbb{C} \oplus \mathbb{C} \qquad \left[\begin{array}{l} f(r^j) = z^{-j} f(e) \\ f(sr^j) = z^{j} f(s) \end{array} \right].$$
$$f \longmapsto (f(e), f(s))$$

G-action on V is given by: (easy check)

$$s \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad r \mapsto \begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix}.$$

§3. Frobenius' formula for characters of induced representations..

Theorem.- [G: finite group; $H < G$ a subgroup; $\rho : H \to GL(W)$ a f.d. repn]

$$\boxed{\chi_{\mathrm{Ind}_H^G(W)}(g) = \frac{1}{|H|} \sum_{\substack{x \in G: \\ x^{-1} g x \in H}} \chi_W(x^{-1} g x)}$$

Proof.- Let $g_1, \dots, g_\ell \in G$ be coset representatives in G/H, i.e. $\cancel{G/H}$ $G = \bigsqcup_{j=1}^{\ell} g_j H$. Then we have

$$V = \mathrm{Ind}_H^G(W) = \bigoplus_{j=1}^{\ell} W^{(j)} \quad \text{where}$$

$$W^{(j)} = \{ f \in V : f(g_i h) = 0 \ \forall i \neq j, h \in H \} \subset V.$$

Let $g \in G$ and $f \in W^{(j)}$. For $1 \leq k \leq l$ and $h \in H$,

we have: $(g \cdot f)(g_k h) = f(g^{-1} g_k h) = 0$ if $g^{-1} g_k \notin g_j H$.

$\Rightarrow g \cdot f \in W^{(k)}$ where k is s.t. $g g_j H = g_k H$.

Since trace only depends on diagonal blocks, we get:

$$\text{Trace of } g \text{ acting on } V = \sum_{j \,:\, g g_j H = g_j H} \text{Trace of } g \text{ acting on } W^{(j)}$$

$$= \sum_{j \,:\, g_j^{-1} g g_j \in H} \text{Trace of } g \text{ acting on } W^{(j)}$$

Ex.: Verify that the following diagram commutes:

$$\begin{tikzcd}
W^{(j)} \arrow[r, "\sim", "f \mapsto f(g_j)"'] \arrow[d, "g"'] & W \arrow[d, "g_j^{-1} g g_j"] \\
W^{(j)} \arrow[r, "\sim"] & W
\end{tikzcd}$$

Hence,

$$\chi_{\mathrm{Ind}_H^G(W)}(g) = \sum_{\substack{j \text{ s.t.} \\ g_j^{-1} g g_j \in H}} \chi_W(g_j^{-1} g g_j)$$

Note : $\{ j : g_j^{-1} g g_j \in H \} \longleftrightarrow \{ x \in G : x^{-1} g x \in H \} / H$

(if $x^{-1} g x \in H$ then $(xh)^{-1} g (xh) \in H \ \forall h \in H$)

$$\Rightarrow \quad \chi_{\mathrm{Ind}_H^G (W)} (g) = \frac{1}{|H|} \sum_{\substack{x \in G : \\ x^{-1} g x \in H}} \chi_W (x^{-1} g x)$$

□

§4. Note that Frobenius' formula in fact defines a linear map

$$(\mathbb{C} H)_{\text{class}} \longrightarrow (\mathbb{C} G)_{\text{class}}$$

$$\psi \longmapsto \mathrm{Ind}_H^G (\psi) : g \mapsto \frac{1}{|H|} \sum_{\substack{x \in G : \\ x^{-1} g x \in H}} \psi (x^{-1} g x)$$

Our next result shows that induction is dual to restriction w.r.t. the usual inner product on class functions – i.e.,

$$(\phi , \mathrm{Ind}_H^G (\psi))_{\mathbb{C} G_{\text{class}}} = (\mathrm{Res}_H^G \phi , \psi)_{\mathbb{C} H_{\text{class}}}$$

$\forall \ \phi \in \mathbb{C} G_{\text{class}}$
$\psi \in \mathbb{C} H_{\text{class}}$

Theorem.- (Frobenius reciprocity) :

There is a natural isomorphism of vector spaces ; $\forall \begin{array}{l} V \in \mathrm{Rep}(G) \\ W \in \mathrm{Rep}(H) \end{array}$

$$\Phi : \mathrm{Hom}_H\left(\mathrm{Res}^G_H V, W\right) \longrightarrow \mathrm{Hom}_G\left(V, \mathrm{Ind}^G_H(W)\right)$$

given by:

$$\Phi(A)(v) : \begin{array}{rcl} G & \to & W \\ g & \mapsto & A(g^{-1}\cdot v) \end{array}$$

$\forall\ A : V \to W$ H-intertwiner, $v \in V$ and $g \in G$

Proof left as a routine verification.