

§1. Linearization of homogeneous polynomial functors. -

Recall: $\underline{\text{Vect}}$ = category of finite-dim'l \mathbb{C} -vector spaces.

A functor $F: \underline{\text{Vect}} \rightarrow \underline{\text{Vect}}$ is a polynomial functor if $\forall X, Y \in \underline{\text{Vect}}$, $r \geq 1$, and $f_1, \dots, f_r: X \rightarrow Y$ linear maps,

$F(\lambda_1 f_1 + \dots + \lambda_r f_r)$ is a polynomial in $\lambda_1, \dots, \lambda_r$ (with coefficients from $\text{Hom}_{\mathbb{C}}(F(X), F(Y))$.)

F is said to be homogeneous of degree $n \in \mathbb{N}$, if $F(\lambda_1 f_1 + \dots + \lambda_r f_r)$ is hgs. of deg. n in $\lambda_1, \dots, \lambda_r$.

By Lemma §4 of Lecture 17, every poly functor is a direct sum of hgs poly. functors.

Let $n \in \mathbb{Z}_{\geq 1}$ and $\text{Poly}_n =$ category of hgs. poly functors of degree n .

[Objects: $F: \underline{\text{Vect}} \rightarrow \underline{\text{Vect}}$ hgs. poly. deg = n .

Morphisms: $\xi: F_1 \rightarrow F_2$ natural transformations

- i.e. $\xi_V \in \text{Hom}(F_1(V), F_2(V)) \forall V \in \underline{\text{Vect}}$

s.t.
$$\begin{array}{ccc} F_1(V) & \xrightarrow{\xi_V} & F_2(V) \\ F_1(f) \downarrow & & \downarrow F_2(f) \\ F_1(W) & \xrightarrow{\xi_W} & F_2(W) \end{array} \quad \text{commutes} \quad \forall f: V \rightarrow W$$

$F \in \text{Poly}_n \implies L_F: \underline{\text{Vect}}^n \rightarrow \underline{\text{Vect}}$ linear in each argument.
(linearization of F)

$L_F(X_1, \dots, X_n)$ was defined as follows. Consider the decomposition:

$$F(\lambda_1 Id_{X_1} \oplus \dots \oplus \lambda_n Id_{X_n}) = \sum_{\substack{m_1, \dots, m_n \in \mathbb{N} \\ m_1 + \dots + m_n = n}} \lambda_1^{m_1} \dots \lambda_n^{m_n} \underbrace{\left(\underbrace{u_{\underline{m}}(\underline{X})}_{m} \right)}_{\in \text{End}(F(X_1 \oplus \dots \oplus X_n))}$$

As in the proof of Lemma §4, Lecture 17,

complete set of pairwise orthogonal idempotents

$$F(X_1 \oplus \dots \oplus X_n) \simeq \bigoplus_{\substack{m_1, \dots, m_n \\ \text{as above}}} F_{m_1, \dots, m_n}(X_1, \dots, X_n)$$

naturally in X_1, \dots, X_n . $L_F := F_{1, \dots, 1}$

Notation $v(\underline{X}) = u_{1, \dots, 1}(\underline{X}) : F(\bigoplus X_j) \rightarrow F(\bigoplus X_j)$ is the projection onto $L_F(X_1, \dots, X_n)$ summand.

§2. Action of the symmetric group - For every $X \in \underline{\text{Vect}}$, the vector

space $L_F^{(n)}(X) := L_F(X, \dots, X)$ carries a natural (in X) action of the symmetric group S_n . To describe it, let us

introduce some notations.

$$Y = \underbrace{X \oplus \dots \oplus X}_{n \text{ copies}} \quad \forall 1 \leq \alpha \leq n, \quad \begin{matrix} i_\alpha : X \rightarrow Y \\ p_\alpha : Y \rightarrow X \end{matrix} \begin{matrix} \text{canonical} \\ \text{inclusion} \\ \text{\& projection} \end{matrix}$$

Note: $p_\beta \circ i_\alpha = \delta_{\alpha\beta} \cdot Id_X$ $\sum_\alpha i_\alpha p_\alpha = Id_Y$

For $w \in S_n$, let $\sigma_w : Y \rightarrow Y$ be given by (recall $Y = X^{\oplus n}$)

$$\sigma_w = \sum_{\alpha=1}^n i_{w(\alpha)} \circ p_\alpha \quad [\sigma_w \text{ simply permutes the direct summands of } X^{\oplus n}.]$$

Therefore, we get $F(\sigma_w) \in F(Y)$.

Note: for $(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$, if $D_\lambda = \sum_{\alpha=1}^n \lambda_\alpha \text{Id}_X = \sum_{\alpha=1}^n \lambda_\alpha i_\alpha p_\alpha \in \text{End}(Y)$

then $\boxed{\sigma_w \circ D_\lambda = D_{w\lambda} \circ \sigma_w}$ $w\lambda = (\lambda_{w^{-1}(1)}, \dots, \lambda_{w^{-1}(n)})$.

(Proof: - $\sigma_w \circ D_\lambda = \sum_{\alpha, \beta} i_{w(\alpha)} p_\alpha \lambda_\beta i_\beta p_\beta$
 $= \sum_{\alpha} \lambda_\alpha i_{w(\alpha)} p_\alpha$ ($p_\alpha i_\beta = \delta_{\alpha\beta} \text{Id}$)

$D_{w\lambda} \circ \sigma_w = \sum_{\alpha, \beta} \lambda_{w^{-1}(\alpha)} i_\alpha p_\alpha i_{w(\beta)} p_\beta$
 $= \sum_{\beta} \lambda_\beta i_{w(\beta)} p_\beta$ ($= \begin{cases} \text{Id} & \text{for } \alpha = w\beta \\ 0 & \text{o/w} \end{cases}$)

Applying F gives $\boxed{F(\sigma_w) \circ F(D_\lambda) = F(D_{w\lambda}) \circ F(\sigma_w)}$

Taking coefficient of $\lambda_1 \dots \lambda_n$ on both sides implies that $F(\sigma_w)$ preserves the summand $L_F^{(n)}(X)$ of $F(Y)$.

i.e. $F(\sigma_w)$ commutes with ν .

Let $F'(\sigma_w)$ denote this restriction. That is, if $j_X : L_F^{(n)}(X) \hookrightarrow F(Y)$
 $q_X : F(Y) \twoheadrightarrow L_F^{(n)}(X)$
are natural inclusions & projection

then, $\boxed{F'(\sigma_w) = q_X \circ F(\sigma_w) \circ j_X}$

$(q \circ j = \text{Id on } L_F^{(n)}(X))$
 $(j \circ q = \nu)$

In conclusion, $w \mapsto F'(\sigma_w)$ gives a natural S_n -action on $L_F^{(n)}(X)$. (Easy exercise - check $F'(\sigma_{w_1 w_2}) = F'(\sigma_{w_1}) F'(\sigma_{w_2})$.)

§3. Recovering F from $L_F^{(n)}$ -- ($X \in \underline{Vect}$; $Y = X^{\oplus n}$)

Let $i : X \rightarrow Y$ and $p : Y \rightarrow X$ be defined by $i_\alpha : X \rightarrow Y$ $p_\alpha : Y \rightarrow X$ as before

$i = \sum_\alpha i_\alpha$; $p = \sum_\alpha p_\alpha$. Consider the natural transformations

$\xi = q \circ F(i) : F \rightarrow L_F^{(n)}$ (recall: $L_F^{(n)}(X) \xrightarrow{j} F(Y) \xleftarrow{q}$)

$\eta = F(p) \circ j : L_F^{(n)} \rightarrow F$

Theorem. - $\eta \circ \xi = \text{mult. by } n!$
 $\xi \circ \eta = \sum_{w \in S_n} F'(\sigma_w)$ as defined in §2 above.

Proof. - $\eta \circ \xi = F(p) \circ j \circ q \circ F(i) = F(p) \circ v \circ F(i)$.

Thus, for a given $X \in \underline{Vect}$ ($Y = X^{\oplus n}$), $(\eta \circ \xi)_X$ is the coefficient of $\lambda_1 \dots \lambda_n$ in $F(p) \circ F(D_\lambda) \circ F(i)$

$= F(p \circ D_\lambda \circ i)$ [$D_\lambda = \sum_\alpha \lambda_\alpha i_\alpha p_\alpha$]

Now $p \circ D_\lambda \circ i = \text{scalar mult. by } \lambda_1 + \dots + \lambda_n$

$\Rightarrow F(p \circ D_\lambda \circ i) = (\lambda_1 + \dots + \lambda_n)^n \cdot \text{Id}_{F(X)}$

coeff. of $\lambda_1 \dots \lambda_n$ in $(\lambda_1 + \dots + \lambda_n)^n = n!$

For the converse, $\xi \circ \eta = g \circ F(i \circ p) \circ j$

Claim. - $v \circ F(i \circ p) \circ v = \sum_{w \in S_n} F(\sigma_w) \circ v$

Given the claim, we have: $j \circ (\xi \circ \eta) \circ g = v \circ F(i \circ p) \circ v$
($v = j \circ g$)

$$= \sum_{w \in S_n} F(\sigma_w) \circ v$$

As $g \circ j = \text{Id}$, we get $\xi \circ \eta = \sum_{w \in S_n} g \circ F(\sigma_w) \circ j = \sum_{w \in S_n} F'(\sigma_w)$.

Proof of the claim. - Consider linear transformations of the form

$\sum_{\alpha, \beta} f_{\alpha\beta} i_\alpha p_\beta : Y \rightarrow Y$. Thus $F(\sum f_{\alpha\beta} i_\alpha p_\beta)$ will be a polynomial of degree n in n^2 variables $\{f_{\alpha\beta}\}$. For $w \in S_n$, let $c_w \in \text{End}(Y)$ be the coefficient of $f_{w(1),1} \cdots f_{w(n),n}$ in $F(\sum f_{\alpha\beta} i_\alpha p_\beta)$.

Now $F(\sigma_w) \circ v = v \circ F(\sigma_w) \circ v$ as $v^2 = v$ and $F(\sigma_w)$ commutes with v .

is, by definition, the coefficient of $\lambda_1 \cdots \lambda_n \mu_1 \cdots \mu_n$ in

$$F(D_\lambda) F(\sigma_w) F(D_\mu) = F(D_\lambda \circ \sigma_w \circ D_\mu)$$

$$= \sum_{\alpha} \lambda_{w(\alpha)} \mu_\alpha i_{w(\alpha)} p_\alpha$$

Hence, $F(\sigma_w) \circ v = c_w$ as above.

On the other hand, $v \circ F(i \circ p) \circ v$ is the coefficient of $\lambda_1 \cdots \lambda_n \mu_1 \cdots \mu_n$

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$$\text{in } F(D_\lambda \text{ i p } D_\mu) = F\left(\sum_{\alpha, \beta} \lambda_\alpha \mu_\beta i_\alpha p_\beta\right)$$

which is $\sum_{w \in S_n} c_w$. The claim is proved. \square

§4. Corollary. - We have functorial isomorphisms:

$$F(X) \cong L_F^{(n)}(X)^{S_n}$$

(The averaging operator $\frac{1}{n!} \sum_{w \in S_n} w = (n!)^{-1} \xi \circ \eta$

is idempotent, with image = $L_F^{(n)}(X)^{S_n}$.)

Hence $F \in \text{Polyn}$ is of the form $X \mapsto L(X, \dots, X)^{S_n}$ where

$L: \text{Vect}^n \rightarrow \text{Vect}$ is hgs of degree 1 in each variable.

§5. Theorem. Let $L: \text{Vect}^n \rightarrow \text{Vect}$ be n -multi-linear. Then

we have functorial iso.

$$L(X_1, \dots, X_n) \cong \overbrace{L(\mathbb{C}, \dots, \mathbb{C})}^{L^{(n)}(\mathbb{C})} \otimes X_1 \otimes \dots \otimes X_n.$$

Proof. - ($n=1$ case): $L: \text{Vect} \rightarrow \text{Vect}$ is linear, hence

$$L(f_1 + f_2) = L(f_1) + L(f_2) \quad \forall f_1, f_2: X_1 \rightarrow X_2 \text{ in } \text{Vect}.$$

For $x \in X$, let $e_x: \mathbb{C} \rightarrow X$. Consider the natural map

$$\psi_x: \text{Hom}_{\mathbb{C}}(L(X), Y) \rightarrow \text{Hom}_{\mathbb{C}}(X, \text{Hom}_{\mathbb{C}}(L(\mathbb{C}), Y))$$

$$(\psi_x f)(x) = f \circ L(e_x).$$

Combining with tensor-hom adjointness, we have

$$\phi_X : \text{Hom}_{\mathbb{C}}(L(X), Y) \rightarrow \text{Hom}_{\mathbb{C}}(L(\mathbb{C}) \otimes X, Y)$$

We claim that ϕ_X is an isomorphism. Since L is additive, if ϕ_{X_1} and ϕ_{X_2} are iso.'s then so is $\phi_{X_1 \oplus X_2}$. Hence it is enough to show that $\phi_{\mathbb{C}}$ is an iso, which is obvious.

Now we can inductively argue ($n \geq 1$):

$$\begin{aligned} L(X_1, \dots, X_n) &\cong L(X_1, \dots, X_{n-1}, \mathbb{C}) \otimes X_n \\ \dots &\cong L(\mathbb{C}, \dots, \mathbb{C}) \otimes X_1 \otimes \dots \otimes X_n. \end{aligned}$$

□

Conclusion.- $\forall F \in \text{Poly}_n$, we have functorial iso.'s

$$F(X) \cong \left(L_F^{(n)}(\mathbb{C}) \otimes T^n(X) \right)^{S_n}$$