

Recall that we proved the following results about polynomial functors:

- (1) Every poly. functor splits naturally as a direct sum of homogeneous poly. functors.

Let Poly_n = category of hgs. poly. functors of degree n .

$\forall F \in \text{Poly}_n$, we defined $L_F: \text{Vect}^n \rightarrow \text{Vect}$ as the $(1, \dots, 1)$ component of $F(X_1 \oplus \dots \oplus X_n)$. $L_F^{(n)}(X) := L_F(X, \dots, X)$.

- (2) We have natural transformations

$$\xi = \eta \circ F(i) : F \rightarrow L_F^{(n)}$$

$$\eta = F(p) \circ j : L_F^{(n)} \rightarrow F$$

$$\left[\begin{array}{l} \text{here } L_F^{(n)}(X) \begin{array}{c} \xrightarrow{i_X} \\ \xleftarrow{q_X} \end{array} F(X^{\oplus n}) \\ i_X: X \rightarrow X^{\oplus n} \quad p_X: X^{\oplus n} \rightarrow X \\ i_X(x) = (x, \dots, x) \quad \forall x \in X \\ p_X(x_1, \dots, x_n) = \sum_{j=1}^n x_j \end{array} \right]$$

(Thm §3 Lecture 18)

$$\eta \circ \xi = \text{mult. by } n!$$

$$\xi \circ \eta = \sum_{w \in S_n} F'(\sigma_w)$$

$$F'(\sigma_w) \hookrightarrow L_F^{(n)} \quad \text{natural } S_n\text{-action.}$$

- (3) (Cor §4 Lecture 18) We have natural iso.

$$F(X) \cong L_F^{(n)}(X)^{S_n}$$

- (4) Every n -linear $L: \text{Vect}^n \rightarrow \text{Vect}$ is of the form (Thm §5 L18)

$$L(X_1, \dots, X_n) = U \otimes X_1 \otimes \dots \otimes X_n$$

$$\text{where } U = L(\mathbb{C}, \dots, \mathbb{C}) \quad \left(\begin{array}{l} \text{if } L = L_F, \\ U = L_F^{(n)}(\mathbb{C}) \text{ is an } S_n\text{-repr.} \end{array} \right)$$

§1. Theorem. The following are inverse functors to each other:

$$\begin{aligned} \varphi: \text{Poly}_n &\rightarrow S_n\text{-reps} & \psi: S_n\text{-reps} &\rightarrow \text{Poly}_n \\ F &\mapsto L_F^{(n)}(\mathbb{C}) & M &\mapsto \psi(M): X \mapsto (M \otimes X^{\otimes n})^{S_n} \end{aligned}$$

Proof. - $(\psi \circ \varphi)(F): X \mapsto (L_F^{(n)}(\mathbb{C}) \otimes X^{\otimes n})^{S_n}$
 is naturally iso. to F . (see Page 7, L 18).

For the converse, let $M \in S_n\text{-reps}$.

$F = \psi(M): \text{Vect} \rightarrow \text{Vect}$ as above. We compute

its linearization:

$$\begin{aligned} L_F(X_1, \dots, X_n) &= (1, \dots, 1) \text{ component in} \\ F\left(\bigoplus_{j=1}^n X_j\right) &= \left(M \otimes \left(\bigoplus_{j=1}^n X_j\right)^{\otimes n}\right)^{S_n} \\ &= \left(M \otimes \left[\bigoplus_{w \in S_n} X_{w(1)} \otimes \dots \otimes X_{w(n)}\right]\right)^{S_n} \end{aligned}$$

$$\begin{aligned} \Rightarrow \varphi(\psi(M)) &= L_F^{(n)}(\mathbb{C}) \\ &= \left(M \otimes \left[\bigoplus_{w \in S_n} \mathbb{C} \otimes \dots \otimes \mathbb{C}\right]\right)^{S_n} \end{aligned}$$

\uparrow $w(1)$ copy \uparrow $w(n)$ copy in $\underbrace{\mathbb{C} \oplus \dots \oplus \mathbb{C}}_{n\text{-terms}}$

$$= (M \otimes \mathbb{C} S_n)^{S_n}$$

$\cong M$ (easy exercise: $\xi \in (M \otimes \mathbb{C} S_n)^{S_n}$ is completely det. by δ_e -term: $\xi = \sum_{w \in S_n} \xi_w \otimes \delta_w$ in M) □

§2. Trace function. - For $F \in \text{Poly}_n$, $d \in \mathbb{Z}$ and $m \in \mathbb{Z}_{\geq 1}$, (3)

We have $\text{Tr}_{(m)}(F) \in \mathbb{C}[x_1, \dots, x_m]_{\deg=n}^{Sm}$, defined by

$\text{Tr}(F) = \text{Trace of } F(\text{diagonal}(x_1, \dots, x_m)) \text{ acting on } F(\mathbb{C}^m)$

e.g. $F = T^2 : \text{Vect} \rightarrow \text{Vect}$ (hgs. degree 2).

If $\{v_1, \dots, v_m\}$ is the standard basis of \mathbb{C}^m

$D(x) : v_j \mapsto x_j v_j$ diagonal operator. $F(\mathbb{C}^m) = \mathbb{C}^m \otimes \mathbb{C}^m$ has basis $v_i \otimes v_j$ w/ eigenvalue $x_i x_j$
 $i, j \in \{1, \dots, m\}$

$$\text{So, } \text{Tr}(T^2) = \sum_{i,j \in \{1, \dots, m\}} x_i x_j = (x_1 + \dots + x_m)^2$$

$$\text{More generally, } \text{Tr}(T^n) = (x_1 + \dots + x_m)^n$$

Easy exercise: $\text{Tr}(\text{Sym}^n) = h_n(x_1, \dots, x_m)$
complete symm fn.

$\text{Tr}(\Lambda^n) = e_n(x_1, \dots, x_m)$
elementary symm. fn.

[Note $\Lambda^n(V) = 0$ iff $\dim V < n$ $\leftrightarrow e_n(x_1, \dots, x_m) = 0$ iff $m < n$.]

Lemma. $F \in \text{Poly}_n$; $m \in \mathbb{Z}_{\geq 1}$; x_1, \dots, x_m variables. (4)

Then $\text{Trace} (F(D(\underline{x})))$ is $F(\mathbb{C}^m)$ $\hookrightarrow \mathbb{C}^m$
 S_m -invariant (i.e. symmetric in x_1, \dots, x_m).
 $D(\underline{x}) = \text{diagonal} \begin{bmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_m \end{bmatrix}$

Proof. $S_m \curvearrowright \mathbb{C}^m \rightarrow S_m \curvearrowright F(\mathbb{C}^m)$
 permutation action $w \in S_m \rightsquigarrow F(w) \in GL(F(\mathbb{C}^m))$.

Note : $F(D(w\underline{x})) = F(w) F(D(\underline{x})) F(w^{-1})$

$\Rightarrow \text{Tr}(F(D(w\underline{x}))) = \text{Tr}(F(D(\underline{x})))$ i.e., $\text{Tr}(F)$ is

symmetric in x_1, \dots, x_m variables. \square

§3. Prop. (i) $\text{Tr}(F \otimes G) = \text{Tr}(F) \text{Tr}(G)$

(for $F \in \text{Poly}_n$; $G \in \text{Poly}_m$, $F \otimes G \in \text{Poly}_{n+m}$
 $(F \otimes G)(x) = F(x) \otimes G(x)$.)

(ii) $\text{Tr}(F_1 \oplus F_2) = \text{Tr}(F_1) + \text{Tr}(F_2)$

[same proof as for group characters.]

§4. Tensor vs. induction product.

Theorem. - $\text{Poly}_n \times \text{Poly}_m \xrightarrow{\otimes} \text{Poly}_{n+m}$

$(\varphi, \psi) \downarrow$ $S_n\text{-reps} \times S_m\text{-reps} \xrightarrow{*} S_{n+m}\text{-reps}$
 $\downarrow \varphi$

$V * W := \text{Ind}_{S_n \times S_m}^{S_{n+m}} V \otimes W$ (induction product)

commutes, where

Proof.- In light of Thm §1 above, it suffices to show that $\forall V \in \text{Rep}(S_n), W \in \text{Rep}(S_m)$, the following two functors are naturally iso.

$$X \mapsto (V \otimes T^n X)^{S_n} \otimes (W \otimes T^m X)^{S_m} \quad : \quad \psi(V) \otimes \psi(W)$$

[ψ from Thm §1]

$$X \mapsto \cancel{V \otimes W} (V * W \otimes T^{n+m} X)^{S_{n+m}}$$

(i) $\left[\begin{array}{l} \text{For } G \supset A \\ H \supset B \end{array} ; A^G \otimes B^H \cong (A \otimes B)^{G \times H} \text{ - check} \right]$

So, $(V \otimes T^n X)^{S_n} \otimes (W \otimes T^m X)^{S_m}$

$$\cong (V \otimes T^n X \otimes W \otimes T^m X)^{S_n \times S_m}$$

$$\cong (V \otimes W \otimes T^{n+m} X)^{S_n \times S_m}$$

(ii) $\left[\begin{array}{l} H < G, H \supset A, G \supset B : \\ (Ind_H^G(A) \otimes B)^G \cong (A \otimes Res_H^G(B))^H \end{array} \right]$

Hence $(Ind_{S_n \times S_m}^{S_{n+m}} (V \otimes W) \otimes T^{n+m} X)^{S_{n+m}}$

$$\cong (V \otimes W \otimes T^{n+m} X)^{S_n \times S_m} \quad \square$$

§5. Corollary. - Let $\{V_\lambda\}_{\lambda \vdash n} = \text{Irred}_{f.d.}(S_n)$ as in

Frobenius' labelling. (so $\text{Ch}(\chi_{V_\lambda}) = S_\lambda(x_1, \dots, x_m)$
Schur poly.)

Let $\{S^\lambda: \text{Vect} \rightarrow \text{Vect}\}_{\lambda \vdash n}$ be corresponding (under Thm §1 iso.)

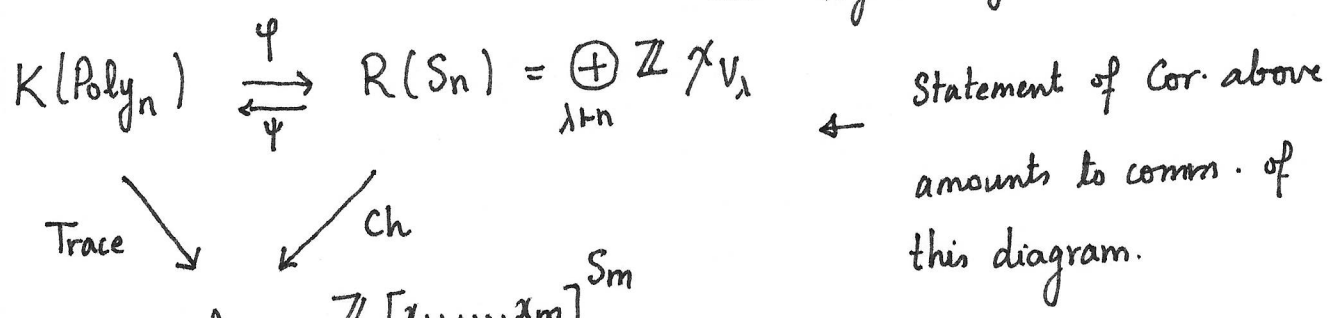
hgs, degree n, poly. functors - called Schur functors. Then

$$\text{Tr}(S^\lambda) = S_\lambda(x_1, \dots, x_m).$$

Remark. - Let $K(\text{Poly}_n) = \text{free ab. gp. on iso. classes of hgs, degree n poly. functors.}$

$$= \bigoplus_{\lambda \vdash n} \mathbb{Z} [S^\lambda]$$

↑ symbol for Schur functor.



Proof. -

We check this diagram commutes for a different \mathbb{Z} -basis of $R(S_n)$

- namely $U_\lambda = \text{Ind}_{S_{\lambda_1} \times \dots \times S_{\lambda_\ell}}^{S_n} (\text{Trivial})$. $\text{Ch}(\chi_{U_\lambda}) = h_\lambda = h_{\lambda_1} \dots h_{\lambda_\ell}$

By Thm §4, $\psi(U_\lambda) \cong \psi(\text{Trivial repn of } S_{\lambda_1}) \otimes \dots \otimes \psi(\text{Triv. of } S_{\lambda_\ell})$

Ex. $\psi(\text{Triv. of } S_\ell) = \text{Sym}^\ell: \text{Vect} \rightarrow \text{Vect}$

and $\text{Trace}(\text{Sym}^\ell) = h_\ell$ (see page 3 above). □