

§1. Some special cases of general results proved so far:

$$(a) \quad \text{Polyn} \cong S_n\text{-repns} \quad \left[\begin{array}{c} F \xrightarrow{\varphi} L_F^{(n)}(\mathbb{C}) \\ \left\{ X \mapsto (M \otimes X^{\otimes n})^{S_n} \right\} \xleftarrow{\psi} M \end{array} \right]$$

$$T^n \longleftrightarrow \mathbb{C} S_n$$

(Schur functors) $S^\lambda \longleftrightarrow V_\lambda$ (irred. repns.) $(\lambda \vdash n)$

$$\text{Sym}^{\lambda_1} \otimes \dots \otimes \text{Sym}^{\lambda_\ell} \longleftrightarrow U_\lambda = \text{Ind}_{S_{\lambda_1} \times \dots \times S_{\lambda_\ell}}^{S_n} (\text{Trivial})$$

$$(b) \quad \text{Tr}_{(m)} : K(\text{Polyn}) \rightarrow \mathbb{C}[x_1, \dots, x_m]_{\text{deg} \leq n}^{S_m}$$

$$F \mapsto \text{Trace of } F \left(\begin{bmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_m \end{bmatrix} \right) \text{ on } F(\mathbb{C}^m)$$

$$\text{Then, } \text{Tr}(\psi(M)) = \text{Ch}(\chi_M)$$

(Frobenius characteristic map)

ie., the following diagram commutes:

$$K(\text{Polyn}) \begin{array}{c} \xrightarrow{\varphi} \\ \xleftarrow{\psi} \end{array} R(S_n)$$

$$\text{Tr} \searrow \quad \swarrow \text{Ch}$$

$$\mathbb{Z}[x_1, \dots, x_m]_{\text{deg} \leq n}^{S_m}$$

(Recall: $K(\text{Polyn}) =$ Grothendieck group of the category Polyn
 $=$ abelian group on $[F]$ - iso classes of $F \in \text{Polyn}$
 subject to $[F \oplus G] = [F] + [G]$)

(c) For every $V \in \underline{\text{Vect}}$ (f.d. \mathbb{C} -vector space) of dim m

Trace of $S^\lambda \left(\begin{bmatrix} x_1 & & 0 \\ & \dots & \\ 0 & & x_m \end{bmatrix} \right)$ on $S^\lambda(\mathbb{C}^m) = S_\lambda(x_1, \dots, x_m)$
(Schur poly.)

Note: $S_\lambda(x_1, \dots, x_m) = 0$ iff $m < l(\lambda)$

(By Young's rule - if $m < l(\lambda)$ then there are no semi-standard Young tableaux of shape λ - filled with numbers $1, \dots, m$.)

Hence $S^\lambda(V) = \{0\}$ if and only if $l(\lambda) > \dim(V)$

$\dim(S^\lambda(V)) = S_\lambda(x_1, \dots, x_m) \Big|_{x_1 = \dots = x_m = 1}$
 $= \# \text{ SSYT of shape } \lambda, \text{ with entries from } \{1, \dots, m\}.$

e.g. $\lambda = (2, 1)$.

$\dim(S^{(2,1)}(V)) = 2 \binom{m}{2} + 2 \binom{m}{3}$
 $= 2 \binom{m+1}{3}$

§2. Decomposition of T^n : As $T^n \leftrightarrow \mathbb{C}S_n$ under $\text{Poly}_n \cong S_n\text{-reps}$,

and $\mathbb{C}S_n \cong \bigoplus_{\lambda \vdash n} V_\lambda^{\oplus d_\lambda}$ ($d_\lambda = \dim V_\lambda$)

We get: $T^n(V) \cong \bigoplus_{\lambda \vdash n} S^\lambda(V)^{\oplus d_\lambda}$

Remarks. - (i) Since each S^λ is a functor, $S^\lambda(V)$ is naturally a repr. of $GL(V)$:

$$g \in GL(V) \rightsquigarrow S^\lambda(g) \in GL(S^\lambda(V)).$$

Hence, the decomposition $T^n(V) = \bigoplus_{\lambda \vdash n} S^\lambda(V)^{\oplus d_\lambda}$ is of $GL(V)$ -reps.

(ii) Viewing $T^n(V)$ as a repr. of S_n , we have:

$$T^n(V) \cong \bigoplus_{\lambda \vdash n} V_\lambda \otimes \underbrace{\text{Hom}_{S_n}(V_\lambda, T^n(V))}_{\text{multiplicity space}}$$

S_n acts here.

$$\text{Hom}_{S_n}(V_\lambda, X) \cong (V_\lambda^* \otimes X)^{S_n} \cong (V_\lambda \otimes X)^{S_n}$$

(for S_n , w, w^{-1} are conjugate to each other $\forall w \in S_n$.)

$$\Rightarrow \chi_{V_\lambda^*} = \chi_{V_\lambda}; \text{ hence } V_\lambda^* \cong V_\lambda.$$

\Rightarrow The multiplicity space above is same as

$$\begin{aligned} \text{Hom}_{S_n}(V_\lambda, T^n(V)) &\cong (V_\lambda \otimes V^{\otimes n})^{S_n} \\ &= S^\lambda(V). \end{aligned}$$

Combining these observations, we get

$$\boxed{T^n(V) \cong \bigoplus_{\lambda \vdash n} V_\lambda \otimes S^\lambda(V)} \text{ as } S_n \times GL(V)\text{-reps.}$$

We will now show that

- Irred. f.d. $GL(V)$ -reps (polynomial) $\leftrightarrow \{ \mathcal{S}^\lambda(V) : \ell(\lambda) \leq \dim V \}$.
- Category of f.d. polynomial reps. of $GL(V)$ is semisimple.

§3. Polynomial representations of $GL_m(\mathbb{C})$. - Recall that a

representation $\alpha : GL_m(\mathbb{C}) \rightarrow GL_N(\mathbb{C})$ is polynomial if

$\forall 1 \leq a, b \leq N$; $\alpha(g)_{a,b}$ is a polynomial in the entries g_{ij} ($1 \leq i, j \leq m$)
 $g \in GL_m(\mathbb{C})$ ((a,b) -th entry of $\alpha(g)$)

Using the same proof as for Lemma §4 of Lecture 17, we have:

every polynomial repn. of $GL_m(\mathbb{C})$ is a direct sum of homogeneous polynomial reps.

Let $R = \mathbb{C}[y_{ij} : 1 \leq i, j \leq m] = \bigoplus_{n=0}^{\infty} R_n$
 \uparrow hgs. poly. of degree n .

Note: $GL_m(\mathbb{C}) \curvearrowright L$ is f.d. hgs of degree n

\Leftrightarrow Matrix coefficients of L are elements of R_n .

[viewing y_{ij} as a function on $GL_m(\mathbb{C})$; $y_{ij}(g) = g_{ij}$.]

In more detail - let $\{e_a : 1 \leq a \leq N\}$ be a basis of L .
 ($N = \dim L$)

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$\forall 1 \leq a, b \leq N$, let $\alpha_{a,b} : GL_m(\mathbb{C}) \rightarrow \mathbb{C}$ be defined as:

$$\alpha_{a,b}(\sigma) = \text{coefficient of } e_b \text{ in } \sigma^{-1}(e_a).$$

L : hqs of degree $n \iff \alpha_{a,b} \in R_n \forall 1 \leq a, b \leq N$.

§4. Prop. - The linear map $\eta : L \rightarrow R_n \otimes \underline{L}$ is an injective
 $e_a \mapsto \sum_{b=1}^N \alpha_{a,b} \otimes e_b$ (auxiliary space)

$GL_m(\mathbb{C})$ - intertwiner. (L : arbitrary f.d. poly. repr; hqs of degree n).
 [see below for $GL_m(\mathbb{C})$ action on $R_n \otimes L$]

Proof. (Injectivity) If $\eta(v) = 0$, then $\forall b$, the fn.

$\alpha_{v,b} : \sigma \mapsto \text{coefficient of } e_b \text{ in } \sigma^{-1}(v)$ is identically zero

Setting $\sigma = Id \in GL_m(\mathbb{C})$ implies $v = 0$.

$\rightarrow GL_m(\mathbb{C})$ action on $R_n \otimes \underline{L}$ is on the first tensor factor:
 (L is aux./mult. space).

Let $v \in L$, $g \in GL_m(\mathbb{C})$. Then

$$\eta(g \cdot v) = \sum_{b=1}^N \alpha_{gv,b} \otimes e_b. \text{ We have to prove the following -}$$

Claim. - $\alpha_{gv,b} = g \cdot (\alpha_{v,b})$

Proof of the claim. - For every $\sigma \in GL_m(\mathbb{C})$, we have

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$$(g \cdot \alpha_{v,b})(\sigma) = \alpha_{v,b}(g^{-1}\sigma) = \text{coeff. of } e_b \text{ in } (g^{-1}\sigma)^{-1}v.$$

$$\alpha_{gv,b}(\sigma) = \text{coeff. of } e_b \text{ in } \sigma^{-1}(gv).$$

Hence the two are equal. \square

§5. In light of Prop §4, every f.d., hgs degree n repr L of $GL_m(\mathbb{C})$ is a subrepresentation of $R_n^{\oplus \dim L}$. We will now prove that R_n is semisimple and $\{L_\lambda = S^\lambda(\mathbb{C}^m)\}_{\substack{\lambda+n \\ \ell(\lambda) \leq m}}$

is the complete set of irred. f.d. hgs. degree n reprs. of $GL_m(\mathbb{C})$.

Theorem. - Matrix coefficients of $\{L_\lambda\}_{\substack{\lambda+n \\ \ell(\lambda) \leq m}}$ form a basis of R_n .

Proof. - Since $T^n(\mathbb{C}^m) \cong \bigoplus_{\substack{\lambda+n \\ \ell(\lambda) \leq m}} L_\lambda^{\oplus \dim V_\lambda}$ as $GL_m(\mathbb{C})$ -reprs;

matrix coefficients of $T^n(\mathbb{C}^m)$ are in the linear span of matrix coefficients of L_λ 's.

But matrix coefficients of $T^n(\mathbb{C}^m)$ contain all degree n monomials in $\{y_{ij}\}_{1 \leq i,j \leq m}$ variables; which span R_n .

So, it suffices to check that

$$\dim R_n = \sum_{\substack{\lambda \vdash n \\ \ell(\lambda) \leq m}} (\dim L_\lambda)^2.$$

Note:
$$\sum_{\substack{\lambda \vdash n \\ \ell(\lambda) \leq m}} \dim(L_\lambda)^2 = \sum_{\lambda \vdash n} S_\lambda(x_1, \dots, x_m) S_\lambda(y_1, \dots, y_m) \left. \begin{array}{l} x_1 = \dots = x_m = 1 \\ y_1 = \dots = y_m = 1 \end{array} \right\}$$

Recall:
$$\sum_{\lambda \vdash n} S_\lambda(x_1, \dots, x_m) S_\lambda(y_1, \dots, y_m) = \text{degree } n \text{ part of } \prod_{1 \leq i, j \leq m} \frac{1}{1 - x_i y_j}$$

$$\sum_{\lambda \vdash n} S_\lambda(1, \dots, 1) S_\lambda(t, \dots, t) \Big|_{t=1} = \text{coefficient of } t^n \text{ in } \frac{1}{(1-t)^{m^2}}.$$

Ex.
$$\dim R_n = \binom{m^2 + n - 1}{n} = \text{coeff. of } t^n \text{ in } \frac{1}{(1-t)^{m^2}}.$$

□

Ex. If $\{L_1, \dots, L_r\}$ are f.d. reps. of a group G s.t. matrix coefficients of L_j 's are linearly independent, then $\{L_1, \dots, L_r\}$ are pairwise, non-iso., irred. reps.