Lecture 21

Summary of results on polynomial repns. of $GL_m(\mathbb{C})$.

- Schur functors. For $\lambda \vdash n$, let V_λ be the finite-dim'l irred. repn. of S_n.

$$\mathbb{S}^\lambda(V) := \left(V_\lambda \otimes T^n(V)\right)^{S_n} \quad : \underline{Vect} \to \underline{Vect}$$

$$\cong \mathrm{Hom}_{S_n}(V_\lambda, T^n(V))$$

- If $\dim(V) = m$, $x_1, \dots, x_m$ variables., then

$$\boxed{\begin{aligned}\text{Trace of } \mathbb{S}^\lambda\left(\begin{bmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_m\end{bmatrix}\right) \text{ on } \mathbb{S}^\lambda(\mathbb{C}^m) \\ = S_\lambda(x_1, \dots, x_m) \ - \text{ Schur polynomial}\end{aligned}}$$

(see Cor §5, page 6 of Lecture 19)

- $$\boxed{T^n(V) \cong \bigoplus_{\lambda \vdash n} V_\lambda \otimes \mathbb{S}^\lambda(V)}$$ naturally in V

  – Schur-Weyl duality (see §2. of Lecture 20).

- Irreducible polynomial repns of $GL_m(\mathbb{C})$ (f.d.) $= \left\{ L_\lambda := \mathbb{S}^\lambda(\mathbb{C}^m) \;\middle|\; \ell(\lambda) \leq m \right\}$

and every f.d. poly. repn. of $GL_m(\mathbb{C})$ is semisimple (or, completely reducible)

Recall the main steps in our proof of this last result.

- from Lecture 20:

(i) $R := \mathbb{C}[y_{ij} : 1 \leq i,j \leq m] = \bigoplus_{n=0}^{\infty} R_n$.

$GL_m(\mathbb{C}) \circlearrowright R_n$ via $g \cdot y_{ij} = \sum_{k=1}^{m} (g^{-1})_{ik}\, y_{kj}$

(ii) Every f.d. poly repn of GL_m = direct sum of hgs, f.d. poly repns. (Lecture 17, Lemma §4).

(iii) Every hgs degree n, f.d. repn. of GL_m is a subrepn. of $R_n^{\oplus \ell}$ (for $\ell \gg 0$ depending on the repn.) (Prop. §4 of Lecture 20).

Finally $\boxed{R_n \cong \bigoplus_{\substack{\lambda \vdash n \\ \ell(\lambda) \leq m}} L_\lambda \otimes \underline{L_\lambda^*}}$ ← aux./mult. space.

"Peter-Weyl theorem" (see Thm §5 of Lecture 20).

→ In this lecture, we give a different proof of the main theorem:

Theorem. - (a) The category of f.d. poly. repns. of $GL_m(\mathbb{C})$ is semisimple.

(b) Irred. f.d. poly. repns. of $GL_m(\mathbb{C})$ $= \{ L_\lambda = \mathbb{S}^\lambda(\mathbb{C}^m) : \ell(\lambda) \leq m \}$

§1. Double commutant theorem. – Let E be a f.d. vector space, (over $\mathbb{C}$).

$A, B \subset \mathrm{End}_{\mathbb{C}}(E)$ two subalgebras such that

(i) A is semisimple and (ii) $B = \mathrm{End}_A(E)$. Then:

(a) $A = \mathrm{End}_B(E)$, (b) B is semisimple, and

(c) $E = \bigoplus_{i=1}^{N} V_i \otimes W_i$ as a repn. of $A \otimes B$, where

$\mathrm{Irr}_{fd}(A) = \{V_1, \dots, V_N\}$ and $\mathrm{Irr}_{fd}(B) = \{W_1, \dots, W_N\}$.

Proof. – As A is f.d. and semisimple, it has finitely many f.d. irred. repns., say $\mathrm{Irr}_{fd}(A) = \{V_1, \dots, V_N\}$, and:

$$E \cong \bigoplus_{i=1}^{N} V_i \otimes W_i \quad , \text{ where } W_i = \mathrm{Hom}_A(V_i, E).$$

$$A \cong \bigoplus_{i=1}^{N} \mathrm{End}_{\mathbb{C}}(V_i).$$

By Schur's lemma, $B = \mathrm{End}_A(E) \cong \bigoplus_{i=1}^{N} \mathrm{End}_{\mathbb{C}}(W_i)$.

This implies all the statements of the theorem. □

§2. Take $E = T^n(V) = \underbrace{V \otimes \cdots \otimes V}_{n\text{-fold}}$; A = Image of $\mathbb{C}S_n \to \mathrm{End}_{\mathbb{C}}(E)$

Lemma. – $B = \mathrm{End}_A(E)$ is spanned by $\{g^{\otimes n} : g \in GL(V)\}$

Proof. – $\mathrm{End}_{S_n}(V \otimes \cdots \otimes V) \cong (\mathrm{End}(V) \otimes \cdots \otimes \mathrm{End}(V))^{S_n}$

$$\cong \mathrm{Sym}^n(\mathrm{End}(V)).$$

Claim. – For a f.d. vector space U, $\mathrm{Sym}^n(U)$ is spanned by $\{u \otimes \cdots \otimes u : u \in U\}$

(Proof of the claim :– Verify that $\mathrm{Sym}^n(U)$ is an irred. repn of $GL(U)$. The subspace spanned by $\{u^{\otimes n} : u \in U\}$ is a non-zero $GL(U)$-subrepn, hence $= \mathrm{Sym}^n(U)$. $\square$)

Therefore, $B \cong \mathrm{Sym}^n(\mathrm{End}(V))$ is spanned by $\{f^{\otimes n} : f \in \mathrm{End}(V)\}$

Let $f \in \mathrm{End}(V)$. Then for almost ~~but~~ all $t \in \mathbb{C}$, $t \cdot \mathrm{Id} + f \in GL(V)$. So $(t \cdot \mathrm{Id} + f)^{\otimes n} \in \mathrm{Span}\{g^{\otimes n} : g \in GL(V)\}$ for all but finitely many $t \in \mathbb{C}$.

As $\mathrm{Span}\{g^{\otimes n} : g \in GL(V)\} \subset B$ subspace – hence closed – we get $(t + f)^{\otimes n}$ is in this subspace $\forall t$. Thus $\underset{(t=0)}{f^{\otimes n}}$ is in this subspace. We have proved

$$B = \mathrm{Span}\{f^{\otimes n} : f \in \mathrm{End}(V)\} = \mathrm{Span}\{g^{\otimes n} : g \in GL(V)\}.$$

$\square$

§3. Combining Thm §1 and Lemma §2, we get

Theorem. (Schur-Weyl duality). – Let V be m-dim'l $\mathbb{C}$-vector space.

(i) The image of $\mathbb{C}S_n$ (say A) and span of the image of $GL(V)$, say B, in $\mathrm{End}_{\mathbb{C}}(V^{\otimes n})$ are centralizers of each other.

(ii) A and B are semisimple. In particular $V^{\otimes n}$ is a semisimple $GL(V)$ - repn.

(iii) $$V^{\otimes n} \overset{\text{as } S_n \times GL(V)\text{ - repns.}}{=} \bigoplus_{\lambda \vdash n} V_\lambda \otimes L_\lambda \quad . \quad \{V_\lambda\}_{\lambda \vdash n} = Irr_{fd}(S_n)$$

and L_λ's are irred. $GL(V)$-repns (or zero.).

§4. The remainder of the proof of the main theorem stated on page 2 above remains the same - as in the proof of Thm §5 of Lecture 20. Except, we still have to prove that characters of L_λ's are given by Schur polynomials (this was used in the dimension counting argument - see page 7 of Lecture 20).

<u>Definition.</u> - Given a $GL_m(\mathbb{C})$ - repn (f.d.), L, define

$$\chi_L(x_1, \ldots, x_m) = \text{Trace of } \begin{bmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_m \end{bmatrix} \text{ acting on } L.$$

<u>Theorem.</u> - Let L_λ's be as in Theorem §3 above. Then,

$$\chi_{L_\lambda}(x_1, \ldots, x_m) = s_\lambda(x_1, \ldots, x_m) \quad \text{(Schur poly's).}$$

<u>Proof.</u> - The proof of this theorem is based on computing the trace of (w, g) on $V^{\otimes n}$ $\left(\begin{array}{l} w \in S_n \ ; \ V = \mathbb{C}^m \\ g = \text{diagonal matrix} \begin{bmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_m \end{bmatrix} \end{array} \right)$

Let $e_1, \dots, e_m$ be the standard basis of $\mathbb{C}^m$, so $g \cdot e_i = x_i e_i$. $(1 \leq i \leq m)$

Basis of $(\mathbb{C}^m)^{\otimes n}$: $\{ e_{i_1} \otimes \dots \otimes e_{i_n} \mid i_1, \dots, i_n \in \{1, \dots, m\} \}$

with S_n - action : $w \cdot (e_{i_1} \otimes \dots \otimes e_{i_n}) = e_{i_{w(1)}} \otimes \dots \otimes e_{i_{w(n)}}$

and $g \cdot (e_{i_1} \otimes \dots \otimes e_{i_n}) = x_{i_1} \cdots x_{i_n} (e_{i_1} \otimes \dots \otimes e_{i_n})$.

Therefore, as S_n-repns, $(\mathbb{C}^m)^{\otimes n} \cong \mathrm{Fun}(I_{m,n}; \mathbb{C})$ where

$$I_{m,n} = \{ \underline{i} : \{1, \dots, n\} \to \{1, \dots, m\} \}.$$

Note that $I_{m,n} \longleftrightarrow \mathcal{X}_{m,n} = \cancel{\{ (S_1, \dots, S_m) \mid \{1, \dots, n\} = \bigsqcup_{j=1}^m S_j \}}$

$$= \{ (J_1, \dots, J_m) \mid \{1, \dots, n\} = \bigsqcup_{i=1}^m J_i \}$$

$$\underline{i} \longmapsto (J_1, \dots, J_m) \text{ where } J_s = \{ k \mid i_k = s \}$$

$$\underline{i} \longleftarrow (J_1, \dots, J_m)$$

where $i_k = s$ s.t. $k \in J_s$.

Hence, as S_n-repns : $(\mathbb{C}^m)^{\otimes n} \cong \mathrm{Fun}(\mathcal{X}_{m,n}; \mathbb{C})$

Moreover, $\mathcal{X}_{m,n} = \bigsqcup_{\substack{\alpha_1, \dots, \alpha_m \in \mathbb{N} \\ \text{s.t. } \sum \alpha_j = n}} \mathcal{X}_{m,n}(\underline{\alpha})$ as sets with S_n-action

$$\mathcal{X}_{m,n}(\underline{\alpha}) = \{ (J_1, \dots, J_m) \in \mathcal{X}_{m,n} \mid |J_j| = \alpha_j \}$$

and every basis vector of $(\mathbb{C}^m)^{\otimes n}$ corr. to $(J_1, \dots, J_m) \in \mathcal{X}_{m,n}(\underline{\alpha})$, is an eigenvector of g with eigenvalue $x_1^{\alpha_1} \cdots x_m^{\alpha_m}$.

(to see why, $(J_1, \ldots, J_m) \longleftrightarrow \underset{\substack{\uparrow \\ g\text{-eigenvalue}}}{e_{i_1} \otimes \cdots \otimes e_{i_n}}$ where $i_k = s$ s.t. $k \in J_s$.

$$x_{i_1} \cdots x_{i_n} = \prod_{\ell=1}^{m} x_\ell^{|J_\ell|} = x_1^{\alpha_1} \cdots x_m^{\alpha_m}.)$$

Thus, trace of (w, g) on $(\mathbb{C}^m)^{\otimes n}$

$$= \sum_{\substack{\underline{\alpha} \in \mathbb{N}^m: \\ \alpha_1 + \ldots + \alpha_m = n}} \left(\text{Trace of } w \text{ on } \mathrm{Fun}(\mathcal{X}(\alpha); \mathbb{C})\right) \cdot x_1^{\alpha_1} \cdots x_m^{\alpha_m}$$

[using $\mathrm{Fun}(\mathcal{X}(\underline{\alpha}), \mathbb{C}) \cong \mathrm{Fun}(\mathcal{X}(\sigma\underline{\alpha}); \mathbb{C})$ $\forall \sigma \in S_m$.]

$$= \sum_{\substack{\lambda \vdash n \\ \ell(\lambda) \le m}} \left(\text{Trace of } w \text{ on } \underbrace{U_\lambda}_{\text{"partition repn"}}\right) \cdot \underbrace{m_\lambda}_{\text{monomial symm. fn.}}$$

Using $U_\lambda \cong \bigoplus_\mu V_\mu^{\oplus K_{\mu\lambda}}$, we get ($K_{\mu\lambda}$: Kostka numbers - see §1 of Lecture 14).

$$\text{Trace of } (w, g) \text{ on } (\mathbb{C}^m)^{\otimes n} = \sum_{\substack{\lambda \vdash n \\ \ell(\lambda) \le m}} \left(\sum_\mu K_{\mu\lambda} \chi_{V_\mu}(w)\right) m_\lambda$$

$$= \sum_{\mu \vdash n} \chi_{V_\mu}(w) \cdot \boxed{\sum_{\lambda \vdash n} K_{\mu\lambda} m_\lambda}$$

$S_\mu(x_1, \ldots, x_m)$.

(Lecture 14, § 1, 2)

Compare with the answer obtained from Schur-Weyl duality,

$$V^{\otimes n} = \bigoplus_{\mu} \underbrace{V_\mu}_{\substack{w \text{ acts} \\ \text{here}}} \otimes \underbrace{L_\mu}_{\substack{g \text{ acts} \\ \text{here}}}$$

(Thm §3, page 4 above)

– i.e., Trace of (w, g) acting on $(\mathbb{C}^m)^{\otimes n}$

$$= \sum_{\mu \vdash n} \chi_{V_\mu}(w) \cdot \chi_{L_\mu}(x_1, \dots, x_m)$$

we get $\chi_{L_\mu}(x_1, \dots, x_m) = s_\mu(x_1, \dots, x_m)$ as claimed $\square$