

Lie algebras and their representations.

§1. Definition. - A Lie algebra^{*} \mathfrak{g} (over a field k) is a k -vector space, together with a bilinear map $[\cdot, \cdot]: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying (called Lie bracket)

(i) Skew-symmetry: $[X, X] = 0 \quad \forall X \in \mathfrak{g}$.

(or, equivalently - if $\text{char}(k) \neq 2$. $[X, Y] = -[Y, X] \quad \forall X, Y \in \mathfrak{g}$.)

(ii) Jacobi identity: $\forall X, Y, Z \in \mathfrak{g}$, we have:

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0.$$

§2. Main examples. - (i) Derivations - Let A be a k -algebra. (not necessarily comm., assoc. or unital)

$$\text{Der}_k(A) := \left\{ f: A \rightarrow A \text{ } k\text{-linear s.t.} \right. \\ \left. f(ab) = f(a)b + a f(b) \quad \forall a, b \in A \right\}$$

Then $\text{Der}_k(A)$ is a Lie algebra with Lie bracket given by

$$[d_1, d_2] = d_1 \circ d_2 - d_2 \circ d_1.$$

Note: Composition of derivations is not necessarily a derivation, but commutator of two derivations is another derivation.

Proof. - $d_1(d_2(ab)) = d_1(d_2(a)b + a d_2(b))$

$$\left. \begin{array}{l} \text{Sophus Lie (Dec. 17, 1842} \\ \text{- Feb. 18, 1899)} \end{array} \right\} = d_1(d_2(a))b + d_2(a)d_1(b) + d_1(a)d_2(b) \\ + ad_1(d_2(b)) \quad [\forall a, b \in A, d_1, d_2 \in \text{Der}(A)].$$

$$\Rightarrow [d_1, d_2](ab) = ([d_1, d_2](a))b + a \cdot ([d_1, d_2](b)). \quad \square$$

Jacobi identity - $[d_1, [d_2, d_3]] = [d_1, d_2 d_3 - d_3 d_2]$
 $= d_1 d_2 d_3 - d_1 d_3 d_2 - d_2 d_3 d_1 + d_3 d_2 d_1.$

$$\Rightarrow [d_1, [d_2, d_3]] + [d_3, [d_1, d_2]] + [d_2, [d_3, d_1]] = 0.$$

(ii) Assoc. alg's \leadsto Lie alg's : If A is any assoc. alg. / k , then

$L(A) = A$ as vector space, together with

$$[a, b] = ab - ba, \quad \forall a, b \in A,$$

is a Lie algebra.

(iii) $\mathfrak{gl}_N(k) =$ Lie algebra of all $N \times N$ matrices over k .

($A = M_{N \times N}(k)$ in the previous example.)

$\mathfrak{sl}_N(k) \subset \mathfrak{gl}_N(k)$ consisting of all matrices of trace zero,
 is a Lie subalgebra of $\mathfrak{gl}_N(k)$.

(iv) Abelian Lie algebra. - \mathcal{A} is abelian if $[a, b] = 0, \forall a, b \in \mathcal{A}$.

e.g. $\mathfrak{h}_N(k) \subset \mathfrak{gl}_N(k)$ consisting of diagonal matrices

is an abelian Lie algebra.

§2. Homomorphisms of Lie algebras. - Let $(\mathfrak{g}_1, [\cdot, \cdot]_1)$ and

$(\mathfrak{g}_2, [\cdot, \cdot]_2)$ be two Lie algebras. A Lie alg. hom. is a k -linear map

$$f: \mathfrak{g}_1 \rightarrow \mathfrak{g}_2 \quad \text{s.t.} \quad f([x, y]_1) = [f(x), f(y)]_2 \quad \forall x, y \in \mathfrak{g}_1.$$

Example. - Adjoint Hom. - Let $\text{ad}: \mathfrak{g} \rightarrow \text{End}_k(\mathfrak{g})$ be given by

$$\text{ad}(x): y \mapsto [x, y].$$

Jacobi identity can be viewed as a statement that

$$\underline{\text{ad}} \text{ is a Lie alg. hom, and } \underline{\text{ad}(x) \in \text{Der}_k(\mathfrak{g})} \quad \forall x \in \mathfrak{g}.$$

$$\text{(Proof. - } \text{ad}(x) \in \text{Der}_k(\mathfrak{g}) \Leftrightarrow \text{ad}(x)([y, z]) = [\text{ad}(x)(y), z] + [y, \text{ad}(x)(z)]$$

$$\Leftrightarrow [x, [y, z]] = [[x, y], z] + [y, [x, z]] \quad \forall y, z \in \mathfrak{g} \quad \left(\begin{array}{l} \text{Jacobi id-} \\ \text{assuming} \\ \text{skew-symmetry} \end{array} \right)$$

Similarly, ad is a Lie alg. hom.

$$\Leftrightarrow \text{ad}([x, y]) = [\text{ad}(x), \text{ad}(y)] \quad \forall x, y \in \mathfrak{g}$$

matrix commutator

$$\Leftrightarrow \text{ad}([x, y]) \cdot z = \text{ad}(x) \cdot (\text{ad}(y) \cdot z) - \text{ad}(y) \cdot (\text{ad}(x) \cdot z) \quad \forall x, y, z \in \mathfrak{g}.$$

$$\Leftrightarrow [[x, y], z] = [x, [y, z]] - [y, [x, z]] \quad : \text{Jacobi id.} \quad \square$$

Example. - Free Lie alg - Let $A = \{a_j\}_{j \in J}$ be a set.

$\text{Free}_{(LA)}(A)$ is defined by the following universal property:

There is a canonical (set) inclusion map $A \rightarrow \text{Free}(A)$.

For any Lie algebra \mathfrak{g} and a (set) map $A \rightarrow \mathfrak{g}$, $\exists!$

Lie alg. hom. $\text{Free}(A) \rightarrow \mathfrak{g}$ s.t. $\begin{matrix} & A & \\ \swarrow & & \searrow \\ \text{Free}(A) & \rightarrow & \mathfrak{g} \end{matrix}$ commutes.

- i.e., $\text{Hom}_{LA}(\text{Free}(A), \mathfrak{g}) = \text{Hom}_{\text{set}}(A, \mathfrak{g}) \quad \forall \mathfrak{g}$.

(a) If $|A|=1$, then $\text{Free}(A) = \mathbb{k}$ with zero bracket - necessarily.

(b) If $|A|=2$, say $A = \{x, y\}$, then $\text{Free}(A)$ is infinite-dim'l.

$\text{Free}(A) = \mathbb{k}$ -span of all parenthesized ^{non-empty} words in x, y
quotiented by subspace generated by $(w_1, w_2) = -(w_2, w_1)$
and $(w_1, (w_2 w_3)) + (w_3, (w_1 w_2)) + (w_2, (w_3 w_1))$

- e.g. $\text{Free}(A)$ degree 1 part is 2-dim'l - $\{x, y\}$ basis.
- _____ 2 _____ 1-dim'l - $\{xy\}$ basis.
- _____ 3 _____ 2 dim'l - $\{x(xy), y(xy) = -y(yx)\}$

Exercise - What is the dim. of $\text{Free}(A)_{\text{deg} = N}$?

§3. Representations of Lie algebras. - Let \mathfrak{g} be a Lie algebra.

A representation of \mathfrak{g} is a Lie alg. hom $\mathfrak{g} \xrightarrow{\pi_V} \mathfrak{gl}(V)$
where V is a \mathbb{k} -vector space. Often denoted by $\mathfrak{g} \subset V$,
we usually omit π_V for ease of notation. : $X \cdot v = \pi_V(X)(v)$.

The usual notions of kernel/image of \mathfrak{g} -intertwiners, sub/quotient reps, \oplus of 2 reps., irreducible/indecomposable reps - carry over to this setting of Lie algebras.

$\text{Rep}(\mathfrak{g})$ has \otimes and duals - similar to the case of groups.:

Let V_1, V_2 be two \mathfrak{g} -reps. $\pi_j : \mathfrak{g} \rightarrow \mathfrak{gl}(V_j)$ ($j=1,2$).
L.A. homs.

Then $\pi_{V_1 \otimes V_2}(x) := \pi_1(x) \otimes \text{Id}_{V_2} + \text{Id}_{V_1} \otimes \pi_2(x) \quad \forall x \in \mathfrak{g}$

is a Lie alg. hom $\mathfrak{g} \rightarrow \text{End}_k \mathfrak{gl}(V_1 \otimes V_2)$.

Hence $\mathfrak{g} \subset V_1 \otimes V_2$ by $x \cdot (v_1 \otimes v_2) = (x \cdot v_1) \otimes v_2 + v_1 \otimes (x \cdot v_2)$.

Similarly $\mathfrak{g} \subset V \rightsquigarrow \mathfrak{g} \subset V^*$ $(x \cdot \xi)(v) = -\xi(x \cdot v)$
 $\forall \xi \in V^*, x \in \mathfrak{g}, v \in V$.

$\mathfrak{g} \subset \text{Hom}_k(V_1, V_2)$ by $(x \cdot f)(v_1) = x \cdot (f(v_1)) - f(x \cdot v_1)$
 $\forall f: V_1 \rightarrow V_2$; $v_1 \in V_1$,
 k -linear $x \in \mathfrak{g}$.

Ex. $V_1^* \otimes V_2 \rightarrow \text{Hom}_k(V_1, V_2)$ is \mathfrak{g} -intertwiner.

§ 4. Examples. - (Abelian Lie alg's).

(i) Let \mathfrak{a}_N be abelian Lie algebra of dimension N .

Let $\{x_1, \dots, x_N\}$ be a basis of \mathfrak{a}_N . A repr. of \mathfrak{a}_N , is completely given by N commuting matrices.

Assuming k is alg. closed - if $\{X_1 = p(x_1), \dots, X_N = p(x_n)\}$
 (say $k = \mathbb{C}$) $\subset \text{End}(V)$

Then we can find a joint eigenvector
 for X_1, \dots, X_N ; say v_1 . ($\rho: \mathfrak{sl}_N \rightarrow \mathfrak{gl}(V)$ a
 f.d. repr.)

(i.e. $\exists \gamma_1: \mathfrak{sl}_N \rightarrow \mathbb{C}$, $0 \neq v_1 \in V$ s.t. $\rho(x) \cdot v_1 = \gamma_1(x) \cdot v_1$.)
 linear form

Quotient by subrepr $\mathbb{C} \cdot v_1 \subset V$ and repeat to get a basis
 $\{v_1, \dots, v_\ell\}$ of V , and linear forms $\gamma_1, \dots, \gamma_\ell \in \mathfrak{sl}_N^*$ s.t.

$$\rho(x) \cdot v_j = \gamma_j(x) v_j + \sum_{i=1}^{j-1} c_{ij}(x) v_i$$

[~~Finitely many~~] (i.e., commuting family of linear operators on a
 f.d. vector space over alg. closed field, can
 be simultaneously "triangularized.")

Moreover $\text{Irr}_{\text{fd}}(\mathfrak{sl}_N) \leftrightarrow \mathfrak{sl}_N^*$ (all 1-dim'l).

(ii) Let \mathfrak{g} be the Lie alg. generated by x, y subject to $[x, y] = \gamma y$
 ($\gamma \in k \setminus \{0\}$).

Claim - ($\text{Char } k = 0$) - Every ^{f.d.} irred. \mathfrak{g} -repr. is 1-dim'l.
 $k = \bar{k}$

Pf. - Let $\mathfrak{g} \subset V$ be a f.d. irred. \mathfrak{g} -repr. Let $0 \neq v \in V$ be
 an eigenvector for x , with eigenvalue $c \in k$.

Then $\forall l \geq 0$; $y^l \cdot v$ is an eigenvector for x with eigenvalue
 $c + l \cdot \gamma$ (Proof - $[x, y^l] = \sum_{i=1}^l y^{i-1} [x, y] y^{l-i} = \gamma \cdot l \cdot y^l$
 (This calculation is in $\text{End}(V)$)

$$\Rightarrow x \cdot (y^l \cdot v) = y^l \cdot (x \cdot v) + [x, y^l] \cdot v$$

$$= (c + l\lambda) y^l v.$$

Since all these eigenvalues are distinct ($\text{char } k = 0$), by finite-dim of V ,
 $\exists N$ s.t. $y^N v \neq 0$ and $y^n v = 0 \quad \forall n > N$.

$(N \geq 0)$
 $\Rightarrow k \cdot (y^N v) \subset V$ is a subrepr. By irred. V is 1-dim'l \square

§5. Very important example. - $\mathfrak{g} = \mathfrak{sl}_2(k)$ (k alg. closed, $\text{char}(k) = 0$).
 2×2 matrices of trace zero.

Basis of \mathfrak{g} : $h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

Lie bracket : $[h, e] = 2e$; $[h, f] = -2f$, $[e, f] = h$.

(Abstract presentation of \mathfrak{sl}_2 - three generators & three relⁿs.)

Theorem. - $\text{Irr}_{fd}(\mathfrak{sl}_2) \leftrightarrow \mathbb{Z}_{\geq 0}$
 $L_n \leftrightarrow n$ where L_n is $(n+1)$ -dim'l

irred. repr. given explicitly as:

Basis of L_n : $\{v_0, \dots, v_n\}$

\mathfrak{sl}_2 -action:
$$\begin{cases} h v_j = (n - 2j) v_j \\ f v_j = (j+1) v_{j+1} \\ e v_j = (n - j + 1) v_{j-1} \end{cases} \quad (v_{-1} = v_{n+1} = 0)$$

Proof. - Let V be a f.d. irred. repr. of sl_2 .

By argument of Example (ii) §4, there exists $0 \neq v_0 \in V$ s.t. and $\lambda \in k$

$$\begin{aligned} h \cdot v_0 &= \lambda v_0 \\ e \cdot v_0 &= 0 \end{aligned} \quad \text{Define: } v_j = \frac{f^j}{j!} v_0 \quad \forall j \geq 0.$$

$$\text{(so } h \cdot v_j = (\lambda - 2j) v_j \text{)}$$

$$\text{Note: } [e, f^r] = \sum_{i=1}^r f^{i-1} [e, f] f^{r-i} = \sum_{i=1}^r f^{i-1} h f^{r-i}$$

$$\text{as } h f = f h - 2f = f(h-2)$$

$$\text{we get } h \cdot f^{r-i} = f^{r-i} (h - 2(r-i))$$

[all this happens in $\text{End } V$]

$$\begin{aligned} \Rightarrow [e, f^r] &= \sum_{i=1}^r f^{i-1+r-i} (h - 2(r-i)) = r \cdot f^{r-1} h \\ &\quad - r(r-1) f^{r-1} \\ &= r \cdot f^{r-1} (h - r + 1) \end{aligned}$$

$$\text{Hence, } e \cdot v_j = \frac{e f^j}{j!} v_0 = \frac{\cancel{f^j} e + [e, f^j]}{j!} \cdot v_0 \quad (e \cdot v_0 = 0)$$

$$= j \cdot \frac{f^{j-1}}{j!} (\lambda - j + 1) v_0 = (\lambda - j + 1) v_{j-1}$$

As $\dim V < \infty$, let $N \in \mathbb{Z}_{\geq 0}$ be s.t. $v_N \neq 0$ and $v_{N+1} = 0$. Then

$$0 = e \cdot v_{N+1} = (\lambda - N) v_N \Rightarrow \lambda = N \text{ and } \dim V = N + 1$$

$V \cong L_N$ from Thm. \square