

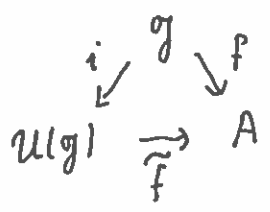
Lecture 24.

Universal enveloping algebras

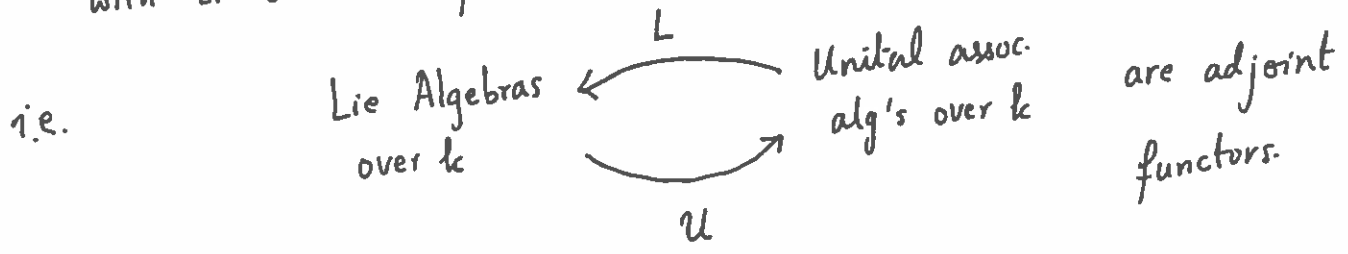
§1. Definition. - Let \mathfrak{g} be a Lie algebra over a field k . The universal enveloping algebra of \mathfrak{g} , denoted by $U(\mathfrak{g})$, is a unital, assoc. alg. / k , satisfying the following universal property.

- There is a k -linear map $\mathfrak{g} \xrightarrow{i} U(\mathfrak{g})$ s.t.

$$i([\mathfrak{x}, \mathfrak{y}]) = i(\mathfrak{x})i(\mathfrak{y}) - i(\mathfrak{y})i(\mathfrak{x}) \quad \forall \mathfrak{x}, \mathfrak{y} \in \mathfrak{g}.$$
- If A is any unital assoc. algebra (over k), and $f: \mathfrak{g} \rightarrow A$ is a k -linear map satisfying $f([\mathfrak{x}, \mathfrak{y}]) = f(\mathfrak{x})f(\mathfrak{y}) - f(\mathfrak{y})f(\mathfrak{x}) \quad \forall \mathfrak{x}, \mathfrak{y} \in \mathfrak{g}$,
 Then $\exists!$ (unital) alg. hom. $\tilde{f}: U(\mathfrak{g}) \rightarrow A$ s.t. the following diagram commutes



In less words, $\text{Hom}_{\text{Lie Alg}}(\mathfrak{g}, L(A)) = \text{Hom}_{k\text{-alg}}(U(\mathfrak{g}), A)$ for every unital, assoc. alg. A . (Recall: $L(A) = A$ as vector space, with Lie bracket defined as commutator).



§2. Construction of $U(\mathfrak{g})$ Given \mathfrak{g} as above, define the tensor algebra of \mathfrak{g} : $T^\bullet(\mathfrak{g}) = \bigoplus_{n=0}^{\infty} T^n(\mathfrak{g})$,

(2)

$T^n(\mathfrak{g}) = \underbrace{\mathfrak{g} \otimes \dots \otimes \mathfrak{g}}_{n\text{-fold}}$. Multiplication in $T^*(\mathfrak{g})$ is nothing but concatenation of tensors.
 $T^0(\mathfrak{g}) = k$

Let $\mathcal{I} \subset T^*(\mathfrak{g})$ be the 2-sided ideal generated by $x \otimes y - y \otimes x - [x, y]$ ($x, y \in \mathfrak{g}$). Then

$$U(\mathfrak{g}) := T^*(\mathfrak{g}) / \mathcal{I}.$$

The map $\mathfrak{g} \rightarrow U(\mathfrak{g})$ arises as the following composition:

$$\mathfrak{g} = T^1(\mathfrak{g}) \hookrightarrow T^*(\mathfrak{g}) \longrightarrow U(\mathfrak{g}) = T^*(\mathfrak{g}) / \mathcal{I}.$$

Verification of the universal property: If A is a unital, assoc. alg.

and $f: \mathfrak{g} \rightarrow A$ is a k -linear map, then - by universal

$$\text{property of } T^*(\mathfrak{g}) \cdot \left[\text{Hom}_{\text{Alg}}(T^*(V), B) = \text{Hom}_{\text{v.s.}}(V, B) \right. \\ \left. \forall B, \text{ unital, assoc. alg.} \right]$$

we get: there is a unique alg. hom. $f_1: T^*(\mathfrak{g}) \rightarrow A$ s.t.
 $f_1(x) = f(x) \forall x \in \mathfrak{g} = T^1(\mathfrak{g})$.

If f satisfies $f([x, y]) = f(x)f(y) - f(y)f(x)$ ($\forall x, y \in \mathfrak{g}$)

then $\mathcal{I} \subset \text{Ker}(f)$. Hence, we have a unique alg. hom

$$\tilde{f}: T^*(\mathfrak{g}) / \mathcal{I} = U(\mathfrak{g}) \longrightarrow A \text{ s.t.}$$

\swarrow
 $U(\mathfrak{g})$

\searrow
 \tilde{f}

\swarrow
 A

\xrightarrow{f}
 commutes.

§3. Remarks and examples. -

(i) Again by universal property of $U(\mathfrak{g})$, we have

$$\text{Hom}_{\text{L.A.}}(\mathfrak{g}, \mathfrak{gl}(V)) = \text{Hom}_{\text{Alg.}}(U(\mathfrak{g}), \text{End}(V))$$

(recall: $\mathfrak{gl}(V) = L(\text{End}V)$).

i.e. $\mathfrak{g}\text{-reps} \leftrightarrow U(\mathfrak{g})\text{-reps.}$

(ii) Gradings and filtrations. - Note: $T^*(\mathfrak{g})$ is \mathbb{N} -graded by:

$$\text{degree of } T^n(\mathfrak{g}) = n \quad \forall n \in \mathbb{N}. \quad (\mathbb{N} = \mathbb{Z}_{\geq 0}).$$

(recall: an \mathbb{N} -graded algebra is an alg. A together with subspaces $A_n \subset A \quad \forall n \in \mathbb{N}$ s.t. $A = \bigoplus_{n \in \mathbb{N}} A_n$ as a vector space,

and $a \in A_n, b \in A_m \Rightarrow a \cdot b \in A_{n+m}$.)

However the ideal $\mathcal{J} = \langle \overbrace{x \otimes y - y \otimes x}^{\text{in deg 2}} - \underbrace{[x, y]}_{\text{in deg 1}} \rangle$ is not homogeneous.

So, $U(\mathfrak{g})$ does not inherit \mathbb{N} -grading from $T^*(\mathfrak{g})$. It only gets a filtration - defined as follows -

$$F_n(U(\mathfrak{g})) := \text{Image of } \left(\bigoplus_{j=0}^n T^j(\mathfrak{g}) \hookrightarrow T^*(\mathfrak{g}) \twoheadrightarrow U(\mathfrak{g}) \right).$$

Check: $F_{-1} = \{0\}; F_0 = k.$

$$F_0 \subset F_1 \subset \dots$$

$$U(\mathfrak{g}) = \bigcup_{n \in \mathbb{N}} F_n(U(\mathfrak{g})).$$

$$a \in F_n, b \in F_m \Rightarrow a \cdot b \in F_{n+m}.$$

Easy exercise : $\bigoplus_{n=0}^{\infty} \mathcal{F}_n / \mathcal{F}_{n-1}$ is a commutative algebra (4)

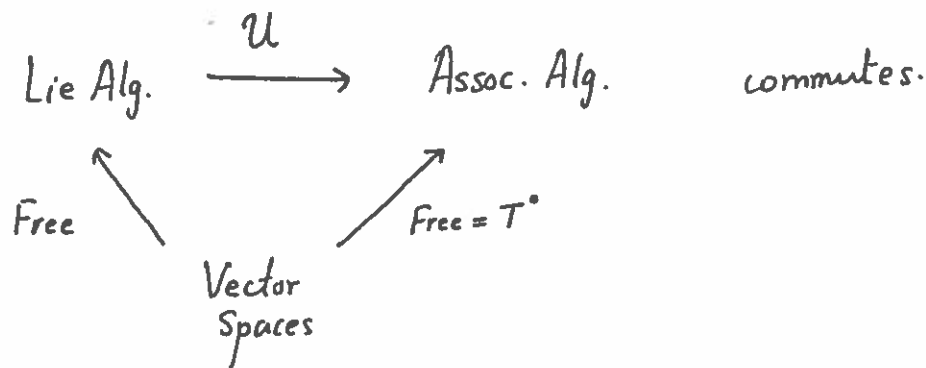
(i.e. given $a \in \mathcal{F}_i$, $b \in \mathcal{F}_j$, $ab - ba \in \mathcal{F}_{i+j-1}$.)

(iii) If $\mathfrak{g} = \text{Free}(V)$ (V : f.d. k -vector space) - defined by

the universal property $\text{Hom}_{\text{LA}}(\text{Free}(V), \mathcal{A}) = \text{Hom}_{\text{v.s.}}(V, \mathcal{A})$ -

(\mathcal{A} : Lie alg. (k)).

then $\mathcal{U}(\mathfrak{g}) \cong T^*(V)$.



(Proof : $\text{Hom}_{\text{Alg}}(\mathcal{U}(\text{Free}(V)), A) = \text{Hom}_{\text{LA}}(\text{Free}(V), L(A))$
 $= \text{Hom}_{\text{v.s.}}(V, A)$
 $= \text{Hom}_{\text{Alg}}(T^*(V), A) \quad \square$)

~~- Application to computing graded dim. of Free Lie alg. (later)~~

(iv) - Generalizing (iii) - if \mathfrak{g} is given by generators and relations - then $\mathcal{U}(\mathfrak{g})$ has the "same" presentation.

e.g. \mathfrak{sl}_2 : generators h, e, f .

$$\text{rel}^n\text{'s: } [h, e] = 2e, \quad [h, f] = -2f \\ [e, f] = h.$$

$U(\mathfrak{sl}_2)$: unital, assoc. alg. generated by h, e, f - subject to relations $he - eh = 2e, \quad hf - fh = -2f, \quad ef - fe = h$.

§4. Poincaré-Birkhoff-Witt (PBW) Theorem. - If \mathfrak{g} has a basis

$\{x_i\}_{i \in I}$ (I : a totally ordered indexing set), then

$U(\mathfrak{g})$ has a basis $\left\{ x_{i_1} x_{i_2} \cdots x_{i_n} : \begin{array}{l} n \geq 0; i_1, \dots, i_n \in I \\ i_1 \leq i_2 \leq \dots \leq i_n \end{array} \right\}$

(Alternately written as: $\bigoplus_{n \in \mathbb{N}} \mathcal{F}_n / \mathcal{F}_{n-1} \cong \text{Sym}^*(\mathfrak{g})$)
(see §3 (ii) for $\mathcal{F}_n(U(\mathfrak{g}))$.)

e.g. $U(\mathfrak{sl}_2)$ has the following basis $\{ f^a h^b e^c : a, b, c \geq 0 \}$.

[A proof of PBW theorem will be presented by Tinghao Huang.]