Lecture 25

Complete reducibility of sl_2-repns.

Recall: $sl_2(\mathbb{C})$ has the following presentation: generators: h, e, f

Lie bracket: $[h,e] = 2e$; $[h,f] = -2f$; $[e,f] = h$.

$\mathrm{Irred}_{fd}(sl_2(\mathbb{C})) = \{L_n : n \in \mathbb{Z}_{\geq 0}\}$ given as:

- L_n has a basis $\{v_0, \ldots, v_n\}$

$h \cdot v_j \cancel{[h, v_j]} = (n-2j)\, v_j$; $e \cdot v_j = (n-j+1)\, v_{j-1}$; $f \cdot v_j = (j+1)\, v_{j+1}$.

§1. Casimir operator. - Given a f.d. repn. $\pi : sl_2 \to gl(V)$,

let $$C_\pi = \frac{\pi(h)^2}{2} + \pi(e)\pi(f) + \pi(f)\pi(e) : V \to V.$$

(or, $C = \frac{h^2}{2} + ef + fe \in \mathcal{U}(sl_2)$.)

Lemma. - C_π is an sl_2-intertwiner.

Pf. - $$[e, C] = \frac{[e,h]\,h + h\,[e,h]}{2} + e\,[e,f] + [e,f]\,e$$
$$= -eh - he + eh + he = 0.$$

Similarly, $[f, C] = 0$ and $[h, C] = 0$. □

Example. $C\big|_{L_n}$ has to be a scalar (by Schur's lemma).

(Casimir element acting on irred. repn. L_n)

and $$C \cdot v_0 = \left(\frac{h^2}{2} + ef + fe\right) \cdot v_0$$

$$= \left(\frac{h^2}{2} + fe + h + fe\right) v_0 \qquad [\, ef = fe + h \text{ in } \mathcal{U}(sl_2)].$$

$$= \left(\frac{n^2}{2} + n\right) v_0 \qquad \text{since} \quad \begin{matrix} e\, v_0 = 0 \\ h\, v_0 = n\, v_0 . \end{matrix}$$

$$= \frac{n^2+2n}{2}\, v_0 .$$

Note - for $m, n \in \mathbb{Z}$; $\quad m^2 + 2m = n^2 + 2n$

$$\Longleftrightarrow (m-n)(m+n+2) = 0$$

i.e. either $m = n$, or $m = -n-2$.

§2.→ Casimir operator allows us to "separate" irred. repns. $\{L_n : n \in \mathbb{Z}_{\geq 0}\}$.

More precisely, let V be an arbitrary f.d. sl_2-repn.

for $l \in \mathbb{C}$, let $V^{(l)} = \{v \in V : (C - l.Id)^N v = 0 \text{ for } N \gg 0\}$

(generalized eigenspace of C w/ eigenvalue l.).

Since C commutes with sl_2-action, each $V^{(l)} \subset V$ is a subrepn

and the generalized eigenspace decomposition $V = \bigoplus_{l \in \mathbb{C}} V^{(l)}$ is a direct

sum dec. of sl_2-repns.

Cor. - If $sl_2 \circlearrowright V$ is a f.d. indecomposable repn, then C has

only one eigenvalue, say $l = \frac{n(n+2)}{2}$ where $n \in \mathbb{Z}_{\geq 0}$ is s.t.

V has a non-zero $v \in V$ s.t. $\begin{matrix} e.v = 0 \\ h.v = nv \end{matrix}$ (every f.d. repn. contains such a vector - see Lecture 23, §4 (ii) page 6 & page 8)

§3. **Theorem.** – $\mathrm{Ext}^1_{sl_2}(L_n, L_m) = \{0\} \quad \forall\, n, m \in \mathbb{Z}_{\geq 0}$.

In simpler terms – if $0 \to L_n \to V \to L_m \to 0$ is a short exact sequence of sl_2-repns., then $V \cong L_n \oplus L_m$ as sl_2-repn.

Proof. – If $n \neq m$, then by previous Cor §2 - ~~eq~~., the generalized eigenspace decomposition of C_V (Casimir element acting on V)

gives $$V \cong \underset{\substack{\uparrow \\ C\text{-eigenvalue} \\ \frac{n(n+2)}{2}}}{L_n} \oplus \underset{\substack{\uparrow \\ \frac{m(m+2)}{2}}}{L_m}$$

and for $n, m \in \mathbb{Z}_{\geq 0}$, $n \neq m \Rightarrow n(n+2) \neq m(m+2)$.

When $n = m$: V has 2-dim'l highest weight space

Let $\{v_0, \dots, v_n\}$ be the usual basis of $L_n \hookrightarrow V$

and $\{w_0, \dots, w_n\} \subset V$ s.t. $\overline{w}_0, \dots, \overline{w}_n \in V/L_n \cong L_n$ is the "usual basis" of L_n.

$V[n] \leftarrow$ span of v_0, w_0

$\vdots$

$V[-n] \leftarrow$ span of v_n, w_n

Exercise. For every $\ell \in \mathbb{Z}_{\geq 0}$, $e^\ell f^\ell = \ell!\, h(h-1)\dots(h-\ell+1)$ on $V[n]$.

Taking $\ell = n+1$ and using the fact that $f^{n+1} \equiv 0$ on $V[n]$, we get that h must have eigenvalues without multiplicity.

As the only eigenvalue of h on $V[n]$ is n, we conclude that $$h = \begin{bmatrix} n & 0 \\ 0 & n \end{bmatrix} \text{ on } V[n] = \mathbb{C}v_0 + \mathbb{C}w_0.$$

So, we obtain $h \cdot v_0 = n v_0$ and $h \cdot w_0 = n w_0$.

Now change w_j's to
$$u_0 = w_0$$
$$u_\ell = \frac{f^\ell}{\ell!} u_0 \quad (0 \le \ell \le n) \; ; \text{ so that}$$

$V_1 = \mathrm{Span}\{u_0, \dots, u_n\} \cong L_n$ and we constructed a section:

$$\begin{tikzcd} 0 \arrow[r] & L_n \arrow[r, "i"] & V \arrow[r, "p"] & L_n \arrow[r] \arrow[l, bend left, "s"] & 0 \end{tikzcd} \qquad s(v_j) = u_j.$$

showing that the sequence splits. □

§4. <u>Corollary.</u> – Every f.d. sl_2-repn. is completely reducible

<u>Proof.</u> – It is enough to show that every short exact sequence
$$0 \to V_1 \to V_2 \to V_3 \to 0, \quad \text{with } V_1 \text{ any f.d. } sl_2\text{-repns}, \; V_3 \text{ irred. f.d.}$$
splits – i.e. $\mathrm{Ext}^1_{sl_2}(V_3, V_1) = \{0\}$ $\forall$ irred. V_3.

So let us fix an irreducible W and argue by induction on the length of a composition series of a f.d. repn. U, that $\mathrm{Ext}^1(W, U) = \{0\}$.

If U is irred. then it follows from Thm §3 above.

If U is not irred, then it has an irred. subrepn $U_1 \subset U$ and $U_2 = U/U_1$ has composition series of smaller length.

by induction $\mathrm{Ext}^1(W, U_1) = \mathrm{Ext}^1(W, U_2) = \{0\}$.

By the long exact sequence (see Problem 10):

$$\underbrace{\mathrm{Ext}^1(W, U_1)}_{= \{0\}} \to \mathrm{Ext}^1(W, U) \to \underbrace{\mathrm{Ext}^1(W, U_2)}_{= \{0\}} \quad \text{is exact}$$

$\Rightarrow \mathrm{Ext}^1(W, U) = \{0\}$. $\square$

§5. Consequences of Lecture 23, Thm §5 and Cor §4 above:

Let $sl_2 \circlearrowright V$ be a f.d. sl_2-repn. Then

(a) h action on V is diagonalizable. i.e.

$$V = \bigoplus_{l \in \mathbb{C}} V[l] \quad \text{where } \underbrace{V[l]}_{\text{(weight space)}} = \{v \in V \mid h.v = lv\}$$

And $V[l] \neq 0 \Rightarrow l \in \mathbb{Z}$ (all weights are integers)

(b) $\forall\, l \in \mathbb{Z}$, $V[l] \cong V[-l]$. (weight symmetry).

(c) If $v \in V[l]$ is s.t. $e.v = 0$, then $f^{l+1}.v = 0$ $(l \geq 0)$.

(d) For $l \in \mathbb{Z}_{\geq 0}$ $\quad V[l] \xrightarrow{e} V[l+2]$ is surjective.

$V[l+2] \xrightarrow{f} V[l]$ is injective