Lecture 28.

Recap:

$$\left[\begin{array}{ll} A : \text{Cartan matrix} ; \; D : \text{symmetrizing integers.} & \\ A \in M_{I\times I}(\mathbb{Z}) \text{ s.t.} & (D = \mathrm{diag}(d_i : i \in I) \in M_{I\times I}(\mathbb{Z})) \quad (d_i \in \mathbb{Z}_{>0}) \\ \quad (i) \; a_{ii} = 2 \; \forall i \in I & (iii) \; d_i a_{ij} = d_j a_{ji} \; \forall i,j \in I \\ \quad (ii) \; a_{ij} \le 0 \; \forall i \neq j & (iv) \; A \text{ is positive-definite.} \\ (\text{normalization: } \gcd(d_i : i \in I) = 1.) & \end{array}\right]$$

To the data of (D, A) we associate: $E = \mathbb{R}\text{-span of } \{h_i : i \in I\}$

$\alpha_j \in E^*$ defined by $\alpha_j(h_i) = a_{ij}$ $\forall i,j \in I$.

$$\phi : E^* \xrightarrow{\sim} E, \quad \alpha_i \longmapsto d_i h_i$$

$(\cdot,\cdot)$ on E^*:

$$(\alpha_i, \alpha_j) = d_i a_{ij}.$$

$(\cdot,\cdot)$ on E – so ϕ is an isometry –

$$(h_i, h_j) = a_{ij} d_j^{-1}.$$

Simple reflections: $s_i(\gamma) = \gamma - \gamma(h_i)\alpha_i \quad (\forall i \in I.) \quad (\gamma \in E^*)$

orthogonal reflection $\left(s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i \quad \forall i,j \in I \right)$

Transported via ϕ: $s_i(x) = x - \alpha_i(x) h_i$ (reflection in $\mathrm{Ker}(\alpha_i) \subset E$).

W = subgroup of $GL(E)$ (or $GL(E^*)$ via ϕ) generated by s_i $(i \in I)$.

$$R = W\text{-orbit of "simple roots" } \{\alpha_i : i \in I\} = \bigcup_{i \in I} W\cdot\alpha_i \subset E^*.$$

Last time we proved that $|R| < \infty$ and $|W| < \infty$.

We also computed all rank 2 examples:

	$A_1 \times A_1$	A_2	B_2	G_2
R	$\{\pm\alpha_1, \pm\alpha_2\}$	$\pm\{\alpha_1, \alpha_2, \alpha_1+\alpha_2\}$	$\pm\left\{\begin{matrix}\alpha_1, \alpha_2, \alpha_1+\alpha_2, \\ 2\alpha_1+\alpha_2\end{matrix}\right\}$	$\pm\left\{\begin{matrix}\alpha_1, \alpha_2, \alpha_1+\alpha_2, \\ 2\alpha_1+\alpha_2, 3\alpha_1+\alpha_2, \\ 3\alpha_1+2\alpha_2\end{matrix}\right\}$
W $= \langle s_1, s_2 \mid s_1^2 = s_2^2 = e$ &	$s_1 s_2 = s_2 s_1$	$s_1 s_2 s_1 = s_2 s_1 s_2$	$s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1$	$s_1 s_2 s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1 s_2 s_1$

<u>"braid rel^{n}s"</u>.

§1. <u>Sign-coherence</u> Let $\widetilde{W}$ = group gen. by σ_i $(i \in I)$ subject to rel^{n}s $\sigma_i^2 = e$ & $(\sigma_i \sigma_j)^{m_{ij}} = e$

where $m_{ij} = 2, 3, 4$ or 6; if

$a_{ij} a_{ji} = 0, 1, 2$ or 3.

Note: $\widetilde{W} \twoheadrightarrow W$, $\sigma_i \mapsto s_i$. We will prove that $\widetilde{W} \simeq W$ via this identification.
(see §4 - pages 7-8 below)

<u>Length function.</u> - For any group G generated by "reflections" - elements of order 2 - $\{g_i\}_{i \in I}$ - define $\ell : G \to \mathbb{Z}_{\geq 0}$ by

$$\ell(g) = \operatorname{Min}\{ k \mid g = g_{i_1} \cdots g_{i_k} \text{ for some } i_1, \dots, i_k \in I \}.$$

(length of g)

Lemma (easy - proof left as an exercise): $\ell(g \cdot g_i) = \ell(g) \pm 1$. $\forall g \in G$, $i \in I$.

Let $R_+ = R \cap \sum_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i$ and $R_- = R \cap \sum_{i \in I} \mathbb{Z}_{\leq 0}\alpha_i$.

<u>Prop.</u> (Sign coherence of roots): -

For every $w \in W$, $i \in I$; if $\ell(ws_i) > \ell(w)$, then $w(\alpha_i) \in R_+$.

(if $\ell(ws_i) < \ell(w)$, then $w(\alpha_i) \in R_-$).

Hence, $R = R_+ \sqcup R_-$ $\quad (R_- = -R_+)$.

<u>Proof</u>. - We will show, by induction on $\ell(w)$, that:

$$ws_i > w \implies w(\alpha_i) \in R_+.$$

Base case: $\ell(w) = 0$ means $w = e$ & the statement is true.

Assume $n = \ell(w) \geq 1$. Pick a reduced expression $w = s_{i_1} \cdots s_{i_n}$
(expression of length $= \ell(w)$)

For convenience, write $j = i_n$. Note: as $ws_i > w$, $i \neq j$.

We choose the smallest expression of w where s_i, s_j are to the right -

\- precisely: let $W_{(i,j)} = \langle s_i, s_j \rangle$

pick v_0 to be of smallest length among

$$\mathcal{P} = \left\{ u \in W \;\middle|\; \begin{array}{l} w = u \cdot r \text{ where } r \in W_{(i,j)} \\ \& \ \ell(w) = \ell(u) + \ell(r) \end{array} \right\}$$

<u>Obs</u>: - • If $u \cdot r$ $(r \in W_{(i,j)})$ is a reduced exp of w; r cannot end with s_i (since $ws_i > w$ is given)

• $w = (s_{i_1} \cdots s_{i_{n-1}})\, \underset{\substack{\parallel \\ s_j}}{s_{i_n}} \implies \ell(v_0) \leq \ell(w) - 1$

- $v_0 s_i > v_0$ and $v_0 s_j > v_0$. Since, otherwise

$$w = v_0 \cdot r_0 = v_0 s_i \cdot s_i r_0 \quad \text{contradict the}$$
$$= v_0 s_j \cdot s_j r_0 \quad \text{minimality of } v_0.$$

By induction, we conclude that $v_0(\alpha_i) \in R_+$ and $v_0(\alpha_j) \in R_+$.

Moreover $w = v_0 \cdot r_0$ where $r_0 = \dots s_i s_j$ reduced word in rank 2.
& cannot be written as $\dots s_j s_i$ (as $ws_i > w$)

So $r_0(\alpha_i) \in \mathbb{Z}_{\geq 0}\alpha_i + \mathbb{Z}_{\geq 0}\alpha_j$
from rank 2 computation. ■

For $R_- = -R_+$, note that $S_{w \cdot \alpha_i} = w\, s_i\, w^{-1}$ $\forall w \in W$; $i \in I$.

hence, if $\alpha = w(\alpha_i) \in R$; $s_\alpha \in W$ and
$s_\alpha(\alpha) = -\alpha \in R$. □

<ins>Cor.</ins> – $\boxed{s_i(R_+) \subset R_+ \cup \{-\alpha_i\} \ (\text{i.e. } \forall \gamma \in R_+, \text{ either } s_i\gamma \in R_+ \text{ or } \gamma = \alpha_i.)}$

§2. <ins>Properties of Weyl groups.</ins> –

– Geometric action. hyperplane arrangement: $\{H_\alpha = \mathrm{Ker}(\alpha) \subset E\}_{\alpha \in R}$.

(Note: $H_\alpha = H_{-\alpha}$).

let $$E^0 = E \setminus \bigcup_{\alpha \in R} H_\alpha$$

– a disconnected space

$\pi_0(E^0)$:= set of connected components of E^0.

As $w \cdot H_\alpha = H_{w(\alpha)}$; W acts on E° and permutes the connected components.

Cor of Sign-Coherence: (Fundamental Chamber) $C_0 := \{ x \in E : \alpha_i(x) > 0 \ \forall i \in I \}$ is a connected component of E°.

Prop. (i) For every $y \in E^\circ$, $\exists\, w \in W$ s.t. $w(y) \in C_0$.

(ii) If $a \in C_0$ and $w(a) = b \in C_0$, then $w = e$ and $a = b$.

[Hence $W \longrightarrow \pi_0(E^\circ_0)$ is a bijection.] and $\overline{C_0}$ = fundamental domain for $W \circlearrowright E$
$w \longmapsto w(C_0)$

Proof.- (i) Choose $a \in C_0$ and pick $y_0 \in \underbrace{W \cdot y}_{\text{finite set}}$ closest to a.

Claim: $y_0 \in C_0$. If not, then there exists $i \in I$ s.t. $\alpha_i(y_0) < 0$.

We can check that this implies $dist(a, s_i(y_0)) < dist(a; y_0)$ Contradicting the choice of y_0.

(As $s_i(y_0) = y_0 - \alpha_i(y_0)\, h_i$; we get : $(a - s_i(y_0), a - s_i(y_0))$
and $(s_i(y_0), s_i(y_0)) = (y_0, y_0)$

$$= (a,a) + (y_0, y_0) - 2\,(a, y_0 - \alpha_i(y_0) h_i)$$

$$= (a - y_0, a - y_0) + 2\alpha_i(y_0)\,(a, h_i)$$

As $(a, h_i) = \frac{1}{d_i}\alpha_i(a) > 0$, and $\alpha_i(y_0) < 0$; we get
$$\| a - s_i(y_0) \|^2 < \| a - y_0 \|^2 .)$$

(ii) We will argue by induction on $\ell(w)$ again; to show that

$$\left.\begin{array}{l} a \in \overline{C_0} \\ b = w(a) \in \overline{C_0} \end{array}\right\} \Rightarrow \quad w = e. \quad \text{and hence } a = b.$$

If $\ell(w) = \ell \geq 1$, write $w = s_{i_1} \cdots s_{i_\ell}$ (a reduced expression)

$$= v \cdot s_{i_\ell}. \qquad (v = s_{i_1} \cdots s_{i_{\ell-1}})$$

Note: $\ell(ws_{i_\ell}) < \ell(w) \Rightarrow w(\alpha_{i_\ell}) \in R_-$ (by Prop §1 above)

as $b \in \overline{C_0}$, $(w(\alpha_{i_\ell}))(b) \leq 0$.

But $(w(\alpha_{i_\ell}))(b) = (w(\alpha_{i_\ell}))(w(a)) = \alpha_{i_\ell}(a) \geq 0$.

$\Rightarrow \alpha_{i_\ell}(a) = 0$ and hence $s_{i_\ell}(a) = a$ proving that

$b = s_{i_1} \cdots s_{i_{\ell-1}}(s_{i_\ell}(a)) = v(a)$ and $\ell(v) < \ell(w)$. □

§3. Exchange property.-

Theorem. - Let $w \in W$ and $i \in I$. Then the following are equivalent:

(a) $\ell(ws_i) < \ell(w)$ (b) $w(\alpha_i) \in R_-$

(c) For every expression $w = s_{i_1} \cdots s_{i_n}$, $\exists\, j$ $(1 \leq j \leq n)$.

s.t. $s_{i_{j+1}} \cdots s_{i_n} s_i = s_{i_j} \cdots s_{i_n}$ [Exchange property]

Proof.- (a) ⇒ (b) already proved in Prop §1.

(c) ⇒ (a) Pick a reduced exp. of $w = s_{i_1} \cdots s_{i_r}$. By [Exchange Prop]

$w = s_{i_1} \cdots s_{i_{j-1}} s_{i_{j+1}} \cdots s_{i_r} s_i$ is another reduced exp. of w.

$\Rightarrow ws_i = s_{i_1} \cdots s_{i_{j-1}} s_{i_{j+1}} \cdots s_{i_r}$ has smaller length than w.

<u>(b) ⇒ (c)</u> : $\quad w = s_{i_1} \cdots s_{i_n} \qquad w(\alpha_i) \in R_-$.

Define $\quad \beta_k = s_{i_k} \cdots s_{i_n}(\alpha_i). \qquad \beta_{n+1} = \alpha_i \in R_+$

$$\beta_1 = w(\alpha_i) \in R_-$$

$\Rightarrow \exists!\, r$ s.t. $\quad \beta_{r+1}, \beta_{r+2}, \ldots, \beta_{n+1} \in R_+$ but $\beta_r \in R_-$.
$(1 \leqslant r \leqslant n)$

$$\Rightarrow \quad \beta_r = s_{i_r}(\beta_{r+1}) \in R_- \quad \Longrightarrow \quad \beta_{r+1} = \alpha_{i_r}.$$
$$\beta_{r+1} \in R_+$$

(see Cor on page 4)

That is, $\quad s_{i_{r+1}} \cdots s_{i_n}(\alpha_i) = \alpha_{i_r}$. $\quad$ [Recall $\; w(\alpha) = \beta \Rightarrow w s_\alpha w^{-1} = s_\beta$]

So $\quad s_{i_{r+1}} \cdots s_{i_n} s_i s_{i_n} \cdots s_{i_{r+1}} = s_{i_r}$

$\Rightarrow \quad s_{i_{r+1}} \cdots s_{i_n} s_i = s_{i_r} s_{i_{r+1}} \cdots s_{i_n}$ $\quad$ as claimed. $\square$

§4. <u>Presentation of Weyl group.</u>

(Jacque Tits)

<u>Theorem.</u> – Let M be a semigroup / monoid and $\{g_i\}_{i \in I} \subset M$ be such that $\forall\, i \neq j$, $\quad \underbrace{g_i g_j g_i \cdots}_{m_{ij}\text{-terms}} = \underbrace{g_j g_i g_i \cdots}_{m_{ij} \text{ terms}}$

[$m_{ij} \in \{2,3,4,6\}$ - see page 2]

Let $w \in W$ and choose a reduced expression $w = s_{i_1} \cdots s_{i_n}$.
Then $\quad g(w) := g_{i_1} \cdots g_{i_n}$ is independent of the choice of the reduced expression.

<u>Cor.</u> (see §1 - page 2) : $\widetilde{W} \twoheadrightarrow W$ is an isomorphism.

(so W has a presentation: $\langle s_i \ (i \in I) \mid \underset{(\forall i)}{s_i^2} = \underset{(i \neq j)}{(s_i s_j)^{m_{ij}}} = e \rangle$).

[<u>Proof of the cor</u> -

$$\widetilde{W} = \langle \sigma_i \ (i \in I) \mid \sigma_i^2 = e \ \forall i \ \ \& \ \ (\sigma_i \sigma_j)^{m_{ij}} = e \ \forall i \neq j \rangle$$

$$\begin{array}{ccc} \sigma_i & & \widetilde{W} \\ \downarrow & & \downarrow \pi \text{ (surjective)} \\ s_i & & W \end{array}$$

Define $g : W \to \widetilde{W}$ by
$$\begin{cases} g(s_i) = \cancel{g_i} \ \sigma_i \ \cancel{(g_i)} \text{ and} \\ \text{for } w \in W; \text{ pick a red. exp. } w = s_{i_1} \cdots s_{i_n} \text{ and set} \\ g(w) = \sigma_{i_1} \cdots \sigma_{i_n} \end{cases}$$

By theorem above, g is well-defined.

Exercise. - Verify that g is a group hom. and a two-sided inverse to $\widetilde{W} \xrightarrow{\pi} W$. $\square$]

<u>Proof of the theorem.</u> - Let $R(w)$ = set of all red. exp.'s of w

$= \{ (i_1, \ldots, i_n) \in I^n : w = s_{i_1} \cdots s_{i_n} \}$

(here $n = \ell(w)$).

<u>To show</u> - (using induction on length) : $\underline{i} = (i_1, \ldots, i_n) \in R(w)$, $\underline{j} = (j_1, \ldots, j_n) \in R(w)$ $\Rightarrow g(\underline{i}) = g(\underline{j})$

where $g(\underline{i}) = g_{i_1} \cdots g_{i_n}$.

<u>Note</u>: The claim is true for $n = 0$ $(w = e)$ and $n = 1$ $R(s_i) = \{i\}$ is singleton.

Assume $n \geq 2$. We will ~~show~~ construct new reduced expressions of w, using exchange property, as follows.

- use $w = s_{j_1} \cdots s_{j_n}$ to get $\underline{i}^{(1)} = (j_1, \ldots, \underset{\text{skipped}}{\hat{j}_r}, \ldots, j_n, i_n) \in R(w)$.

  $\ell(w s_{i_n}) < \ell(w)$ (Thm §3)

  $\rightarrow$ note $g(\underline{i}^{(1)}) = \underbrace{g_{j_1} \cdots \hat{g}_{j_r} \cdots g_{j_n}}_{\text{red. exp. of } w s_{i_n} \text{ of length} < n} \cdot g_{i_n}$

  by induction $g_{j_1} \cdots \hat{g}_{j_r} \cdots g_{j_n} = g_{i_1} \cdots g_{i_{n-1}}$

  $\Rightarrow g(\underline{i}^{(1)}) = g(\underline{i})$.

  $\rightarrow$ If $r \neq 1$, then the same induction argument will show that $g(\underline{i}^{(1)}) = g(\underline{j})$ and we are done.

So, either we will be done by induction, or Exchange property always skips the 1^{st} term

$\underline{i} = (i_1, \ldots, i_n)$ $\quad$ $\underline{i}^{(1)} = (j_2, \ldots, j_n, i_n)$ $\quad$ $\underline{i}^{(2)} = (i_3, \ldots, i_n, j_n, i_n)$

$\underline{j} = (j_1, \ldots, j_n)$ $\quad$ $\underline{j}^{(1)} = (i_2, \ldots, i_n, j_n)$ $\quad$ $\underline{j}^{(2)} = (j_3, \ldots, j_n, i_n, j_n)$

$\underline{i}^{(n)} = (\ldots, j_n, i_n, j_n, i_n) \in R(w)$

...

$\underline{j}^{(n)} = (\ldots, i_n, j_n, i_n, j_n) \in R(w)$

either we stop at some point - due to induction - or the process continues. But $g(\underline{i}^{(n)}) = g(\underline{j}^{(n)})$ by our hypothesis. □