Lecture 29

§1. Lie algebra associated to a Cartan matrix. – Let $A = (a_{ij})_{i,j\in I}$ be a Cartan matrix; $D = \text{diagonal}(d_i : i \in I)$ where $d_i \in \mathbb{Z}_{>0}$ ($\forall i \in I$) and $\gcd(d_i : i\in I) = 1$ so that DA is symmetric. Assume that A is connected.

Define $\tilde{\mathfrak{g}}$ to be the Lie alg. generated by $\{e_i, f_i, h_i\}_{i\in I}$ subject to:

$$[h_i, h_j] = 0 \quad \forall\ i,j \in I. \qquad - (1)$$

$$[h_i, e_j] = a_{ij}\, e_j \quad \text{and} \quad [h_i, f_j] = -a_{ij} f_j. \qquad - (2)$$

$$[e_i, f_j] = \delta_{ij}\, h_i \qquad - (3)$$

Some notations: Let $\boxed{\mathfrak{h} = \mathbb{C}\text{-span of } \{h_i\}_{i\in I}}$. Define the linear forms $\alpha_j \in \mathfrak{h}^*$ by $\boxed{\alpha_j(h_i) = a_{ij}\ \forall\ i,j\in I}$. Then rel^{n}s (1) and (2) can be equivalently written as:

$$[h, h'] = 0 \quad \forall\ h, h' \in \mathfrak{h} \qquad - (1')$$

$$[h, e_j] = \alpha_j(h)\, e_j \quad \text{and} \quad [h, f_j] = -\alpha_j(h) f_j \qquad - (2')$$

So, $\mathfrak{h} \subset \tilde{\mathfrak{g}}$ is an abelian Lie subalgebra. Considering the adjoint action of $\mathfrak{h}$ on $\tilde{\mathfrak{g}}$, define (for $\gamma \in \mathfrak{h}^*$):

$$\tilde{\mathfrak{g}}_\gamma = \{ x \in \tilde{\mathfrak{g}} : [h, x] = \gamma(h) x \ \forall\, h \in \mathfrak{h} \}$$

The rel^n (2) implies $e_i \in \tilde{\mathfrak{g}}_{\alpha_i}$ and $f_i \in \tilde{\mathfrak{g}}_{-\alpha_i}$ ($\forall\ i \in I$).

Since $\{e_i, f_i\}$ generate $\tilde{\mathfrak{g}}$, we conclude that $\tilde{\mathfrak{g}}_\gamma \neq \{0\}$

$$\Rightarrow \gamma \in \sum_{i\in I} \mathbb{Z}\alpha_i \subset \mathfrak{h}^*.$$

Let $Q_+ := \{ \sum_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i \subset Q := \sum_{i\in I} \mathbb{Z}\alpha_i \subset \mathfrak{h}^*$;

and let $\tilde{\mathfrak{n}}_\pm$ = Lie subalgebras of $\tilde{\mathfrak{g}}$ generated by $\{e_i : i\in I\}$ (for +) ; $\{f_i : i\in I\}$ (for −).

§2. Theorem. – (i) $\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}_- \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}_+$ as a vector space.

(ii) $\tilde{\mathfrak{n}}_+$ (resp. $\tilde{\mathfrak{n}}_-$) is a free Lie algebra on $\{e_i : i\in I\}$ (resp. $\{f_i : i\in I\}$).

(iii) There is a Lie alg. auto. $\tilde{\omega}$: $e_i \to f_i$, $f_i \to e_i$ $(\forall i\in I)$, $h \mapsto -h$ $(\forall h\in\mathfrak{h})$.

(iv) $$\tilde{\mathfrak{g}} = \Big(\bigoplus_{\alpha\in Q_+\setminus\{0\}} \tilde{\mathfrak{g}}_{-\alpha}\Big) \oplus \mathfrak{h} \oplus \Big(\bigoplus_{\alpha\in Q_+\setminus\{0\}} \tilde{\mathfrak{g}}_{\alpha}\Big)$$

and $\dim \tilde{\mathfrak{g}}_{\pm\alpha} < \infty$ $\forall\ \alpha\in Q_+\setminus\{0\}$. $\tilde{\mathfrak{g}}_{\pm\alpha} \subset \tilde{\mathfrak{n}}_\pm$ $\forall \alpha\in Q_+\setminus\{0\}$.

(v) If $\mathfrak{a} \subset \tilde{\mathfrak{g}}$ is any ideal (i.e., a vector subspace s.t. $[x,\mathfrak{a}] \subset \mathfrak{a}$ $\forall x\in\tilde{\mathfrak{g}}$), then either $\mathfrak{a}\cap\mathfrak{h} = \{0\}$ or $\mathfrak{a} = \tilde{\mathfrak{g}}$.

(Note : $$\mathfrak{a} = \Big(\bigoplus_{\alpha\in Q_+\setminus\{0\}} (\tilde{\mathfrak{g}}_{-\alpha}\cap\mathfrak{a})\Big) \oplus (\mathfrak{a}\cap\mathfrak{h}) \oplus \Big(\bigoplus_{\alpha\in Q_+} (\mathfrak{a}\cap\tilde{\mathfrak{g}}_{\alpha})\Big).$$)

This result is often attributed to C. Chevalley (1948) and Harish-Chandra (1951). We are following the material from Kac: Infinite dim'l Lie alg's, Ch. 1., Thm 1.2

Claude Chevalley Feb. 11, 1909 – June 28, 1984. Harish Chandra Oct. 11, 1923 – Oct 16, 1983.

Proof of Theorem §2. – (iii) is obvious. In order to obtain (i) and (ii), we construct a "large repn." of $\tilde{\mathfrak{g}}$, as follows:

Let $V = \mathbb{C}$-span of x_i $(i \in I)$ and $T(V)$ be the tensor alg. on V. (So, $T(V)$ = unital assoc. alg / $\mathbb{C}$ of polynomials in non-commuting variables $\{x_i : i \in I\}$). Let $\lambda \in \mathfrak{h}^*$.

Define $\pi_\lambda : \tilde{\mathfrak{g}} \to \mathrm{End}_{\mathbb{C}}(T(V))$ by:

- $\pi_\lambda(h)\cdot 1 = \lambda(h)\cdot 1 \quad \forall h \in \mathfrak{h}$. Inductively on degree:
$$\pi_\lambda(h)\cdot(x_i \otimes a) = -\alpha_i(h)\,(x_i \otimes a) + x_i \otimes \pi_\lambda(h)(a).$$
- $\pi_\lambda(f_i)\cdot a = x_i \otimes a \quad (\forall i \in I)$.
- $\pi_\lambda(e_i)\cdot 1 = 0$ and inductively on degree:
$$\pi_\lambda(e_i)\cdot(x_j \otimes a) = \delta_{ij}\,\pi_\lambda(h_i)\cdot a + x_j \otimes \pi_\lambda(e_i)\cdot a.$$

Exercise. – Verify that π_λ is a Lie alg. hom $\tilde{\mathfrak{g}} \to \mathfrak{gl}(T(V))$; i.e. defines a repn. of $\tilde{\mathfrak{g}}$ on $T(V)$.

Proof of (i): From the defining rel^{n}s of $\tilde{\mathfrak{g}}$, it is clear that every $x \in \tilde{\mathfrak{g}}$ can be written as $x_- + x_0 + x_+$ where $x_0 \in \mathfrak{h}$ and $x_\pm \in \tilde{\mathfrak{n}}_\pm$. That is, $\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}_- + \mathfrak{h} + \tilde{\mathfrak{n}}_+$ as vector spaces. To see that it is a direct sum, assume $x_- + x_0 + x_+ = 0$. use
$$\pi_\lambda(x_- + x_0 + x_+)\cdot 1 = \lambda(x_0)\cdot 1 + \pi_\lambda(x_-)\cdot 1 = 0$$
to get $\lambda(x_0) = 0 \ \forall \lambda \in \mathfrak{h}^* \Rightarrow x_0 = 0$. We will see below (part of proof of (ii)) that $u \mapsto \pi_\lambda(u)\cdot 1$ is injective linear map $\tilde{\mathfrak{n}}_- \to T(V)$. So $x_- = 0$

Hence $x_+ = 0$; i.e., $x_- + x_0 + x_+ = 0 \Rightarrow x_+ = x_- = x_0 = 0$.

(i) is proved

Proof of (ii) : Restrict π_λ to $\tilde{\mathfrak{n}}_-$ to get a Lie alg. hom. $\pi : \tilde{\mathfrak{n}}_- \to \mathfrak{gl}(T(V))$.

By the universal property of enveloping alg. $\mathcal{U}(\tilde{\mathfrak{n}}_-)$, we have

$$\begin{tikzcd} \tilde{\mathfrak{n}}_- \arrow[r] \arrow[d, hook] & \mathfrak{gl}(T(V)) \arrow[d, equal] \\ \mathcal{U}(\tilde{\mathfrak{n}}_-) \arrow[r, "\pi"'] & \mathrm{End}_{\mathbb{C}}(T(V)) \end{tikzcd}$$

injective, by PBW Theorem.

Note : $\pi(f_{i_1} \cdots f_{i_n}) \cdot 1 = x_{i_1} \cdots x_{i_n} \in T(V)$

$\Rightarrow \mathcal{U}(\tilde{\mathfrak{n}}_-) \cong T(V)$. This implies $\tilde{\mathfrak{n}}_-$ is the free Lie alg. on f_i $(i \in I)$

(and $u \mapsto \pi_\lambda(u) \cdot 1$ is injective — needed in the proof of (i) above.)

The assertion for $\tilde{\mathfrak{n}}_+$ can be obtained easily by using the involution $\tilde{\omega}$ from (iii)

(iv) is now obvious. - Note $\begin{matrix} x \in \tilde{\mathfrak{g}}_\alpha \\ y \in \tilde{\mathfrak{g}}_\beta \end{matrix} \Rightarrow [x,y] \in \tilde{\mathfrak{g}}_{\alpha+\beta}$; since

$$[h, [x,y]] = [[h,x],y] + [x,[h,y]] = (\alpha(h) + \beta(h)) \cdot [x,y].$$

Proof of (v). - Note that every ideal $\mathfrak{a} \subset \tilde{\mathfrak{g}}$ admits triangular dec.

$$\mathfrak{a} = \Big(\bigoplus_{\alpha \in Q_+ \setminus \{0\}} \mathfrak{a}_{-\alpha} \Big) \oplus \mathfrak{h} \oplus \Big(\bigoplus_{\alpha \in Q_+ \setminus \{0\}} \mathfrak{a}_\alpha \Big)$$

where $\mathfrak{a}_\beta := \mathfrak{a} \cap \tilde{\mathfrak{g}}_\beta$ $\forall \beta \in \pm Q_+$.

$(\tilde{\mathfrak{g}}_0 = \mathfrak{h})$. This is a consequence of the following general linear alg. fact

Fact : if $\mathfrak{h} \circlearrowright U$ is an $\underline{\mathfrak{h}\text{-diagonalizable}}$ repn. of $\mathfrak{h}$; i.e.

$$U = \bigoplus_{\beta \in \mathfrak{h}^*} U_\beta \qquad \left(U_\beta := \{ u \in U : h \cdot u = \beta(h) u \ \forall h \in \mathfrak{h} \} \right),$$

and $U' \subset U$ is an $\mathfrak{h}$-sub-repn., then $\sum^{\text{finite}}_{\beta \in \mathfrak{h}^*} u_\beta \in U' \Rightarrow u_\beta \in U' \ \forall \beta$.

(Proof : $\sum_{i=1}^{n} u_{\beta_i} \in U' \Rightarrow \sum_{i=1}^{n} u_{\beta_i} \cdot \beta_i(h)^l \in U' \quad \forall h \in \mathfrak{h}.\ l \geq 0.$

Pick h s.t. $\beta_i(h) \neq 0 \ \forall i = 1, \ldots, n$. The transformation matrix of

$$\left\{ \sum_{i=1}^{n} \beta_i(h)^l u_{\beta_i} \quad : 0 \leq l \leq n-1 \right\}$$ is invertible, showing

that $u_{\beta_1}, \ldots, u_{\beta_n} \in U'$. $\square$)

Assume $\mathfrak{a} \cap \mathfrak{h} \neq \{0\}$. Let $x \in \mathfrak{a} \cap \mathfrak{h}$; $x \neq 0$. Choose $i \in I$ s.t.

$\alpha_i(x) \neq 0$. (recall: $\{\alpha_i\}$ form a basis of $\mathfrak{h}^*$).

Then $e_i = \dfrac{1}{\alpha_i(x)} [x, e_i] \in \mathfrak{a} \Rightarrow h_i = [e_i, f_i] \in \mathfrak{a}$.

Same logic $\Rightarrow \{e_j, f_j, h_j\} \subset \mathfrak{a} \quad \forall j$ s.t. $a_{ij} \neq 0$.

Assuming our Dynkin diagram is connected, we get all the generators, showing that $\mathfrak{a} = \tilde{\mathfrak{g}}$. $\square$

§3. As a consequence of Thm. §2 (v), there is a unique, proper, maximal ideal, say $\mathfrak{r} \subset \tilde{\mathfrak{g}}$. This is because sum of proper ideals $\{\mathfrak{a}_j\}_{j \in J}$ (proper $\Rightarrow \mathfrak{a}_j \cap \mathfrak{h} = \{0\}$) again intersects trivially with $\mathfrak{h}$, and hence is proper.

Define $\quad \mathfrak{g}(A) := \tilde{\mathfrak{g}}/\mathfrak{r}$.

(or just $\mathfrak{g}$ - having fixed A)

<u>Remark.</u> - It is obvious from the construction that $\mathfrak{g}$ is a <u>simple Lie algebra</u> (i.e., $\underset{\text{ideal}}{\mathfrak{a} \subset \mathfrak{g}} \Rightarrow \mathfrak{a} = \{0\}$ or $\mathfrak{a} = \mathfrak{g}$.).

We will show that $\dim(\mathfrak{g}) < \infty$. The famous Cartan-Killing classification states that all f.d. simple Lie alg's arise this way.

§4. <u>Serre relations.</u> - Let $i, j \in I$, $i \neq j$. Consider the following elements:

$$e_{ij} = \mathrm{ad}(e_i)^{1-a_{ij}} e_j \in \tilde{\mathfrak{n}}_+$$

$$f_{ij} = \mathrm{ad}(f_i)^{1-a_{ij}} f_j \in \tilde{\mathfrak{n}}_-.$$

Lemma. - $\quad [f_k, e_{ij}] = 0 \quad \forall\, k \in I.$

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Jean-Pierre Serre Sept. 15, 1926

Proof.- Let us show $[e_k, f_{ij}] = 0$. This follows immediately from rel^n (3) (see page 1) - in case $k \notin \{i,j\}$.

If $k = j$; then $[e_j, (\mathrm{ad}\, f_i)^{1-a_{ij}} \cdot f_j] = (\mathrm{ad}\, f_i)^{1-a_{ij}} [e_j, f_j]$

$$= (\mathrm{ad}\, f_i)^{1-a_{ij}} \cdot h_j \qquad \text{(as } e_j, f_i \text{ commute)}$$

$$= (\mathrm{ad}\, f_i)^{1-a_{ij}} (a_{ji} f_i) = 0 \text{ if } a_{ij} \leq -1 \text{ since } [f_i, f_i] = 0$$

If $a_{ij} = 0$, then $a_{ji} = 0$, so $\mathrm{ad}(e_j) \cdot \mathrm{ad}(f_i)^{1-a_{ij}} \cdot f_j = 0$ in this case as well.

If $k = i$, we can use sl_2-repn. theory to conclude $[e_i, f_{ij}] = 0$, as follows. Consider $sl_2 \xrightarrow{\varphi_i} \mathrm{End}(\tilde{\mathfrak{g}})$.

$$\varphi_i(e) \cdot x = [e_i, x]$$
$$\varphi_i(f) \cdot x = [f_i, x]$$
$$\varphi_i(h) \cdot x = [h_i, x]$$

As $[e_i, f_j] = 0$, $[h_i, f_j] = -a_{ij} f_j$, we get $\varphi_i(e) \cdot f_j = 0$, $\varphi_i(h) \cdot f_j = -a_{ij} f_j$

(Note : $-a_{ij} \in \mathbb{Z}_{\geq 0}$)

$$\Rightarrow \underbrace{\varphi_i(e)\ \varphi_i(f)^{1-a_{ij}}} \cdot f_j$$

$$= \left[\varphi_i(f)^{1-a_{ij}} \cdot \varphi_i(e) + \varphi_i(f)^{-a_{ij}} \left(\varphi_i(h) + a_{ij}\right)\right] \cdot f_j$$

$= 0$. The proof of $[f_k, e_{ij}] = 0$ is similar. $\square$

Cor. - Let $\mathfrak{s}_\pm \subset \tilde{\mathfrak{n}}_\pm$ be ideals (in $\tilde{\mathfrak{n}}_\pm$) generated by $\{e_{ij} : i \neq j \in I\}$ (for $+$) ; and $\{f_{ij} : i \neq j \in I\}$ (for $-$).

Then $\mathfrak{s}_\pm$ are also ideals in $\tilde{\mathfrak{g}}$.

Hence $\mathfrak{s}_- \oplus \mathfrak{s}_+ \subset \mathfrak{r}$ (unique, max'l proper ideal).

As a consequence $e_{ij} = f_{ij} = 0$ in $\mathfrak{g} = \tilde{\mathfrak{g}}/\mathfrak{r}$.

→ Later we will be able to show that $\mathfrak{s}_- \oplus \mathfrak{s}_+ = \mathfrak{r}$; i.e. $\mathfrak{g}$ admits the following presentation: (Chevalley-Serre):

Generators : e_i, f_i, h_i $(i \in I)$

Rel^{n}s : (1) – (3) as before (see page 1) &

(4) : $\mathrm{ad}(e_i)^{1-a_{ij}} e_j = 0 = \mathrm{ad}(f_i)^{1-a_{ij}} f_j$ $(\forall\, i \neq j)$.

(Serre rel^{n}s)