Recall: (Lecture 29, Thm §2, §4):

A : Cartan matrix

$\left(\text{ie., } A = (a_{ij})_{i,j\in I} \in M_{I\times I}(\mathbb{Z}) \right.$

- $a_{ii} = 2 \quad \forall\, i \in I$
- $a_{ij} \in \mathbb{Z}_{\leq 0} \ \forall\, i \neq j$
- $\exists\, d_i \in \mathbb{Z}_{>0}$ s.t. $d_i a_{ij} = d_j a_{ji}$ $\forall i,j$
- A is positive-def.)

$\rightsquigarrow$ $\tilde{\mathfrak{g}}$: Lie alg. gen. by $\{e_i, f_i, h_i\}_{i\in I}$ subject to 3 rel^{n}s:

(1) $[h_i, h_j] = 0 \quad \forall i,j \in I$

(2) $[h_i, e_j] = a_{ij} e_j$
$\quad\ [h_i, f_j] = -a_{ij} f_j$

(3) $[e_i, f_j] = \delta_{ij} h_i$.

→ $\mathfrak{h} := \bigoplus_{i\in I} \mathbb{C} h_i \subset \tilde{\mathfrak{g}}$
(Cartan subalg.)

→ $\alpha_j \in \mathfrak{h}^*$ defined by $\alpha_j(h_i) = a_{ij} \ \forall i \in I$.

$Q_+ := \sum_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i \subset Q = \sum_{i\in I} \mathbb{Z}\alpha_i \subset \mathfrak{h}^*$
"root lattice"

- $\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}_- \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}_+$ as vector space.
- $\tilde{\mathfrak{n}}_+$ = Free Lie alg. on $\{e_i\}_{i\in I}$ $= \bigoplus_{\alpha\in Q_+\setminus\{0\}} \tilde{\mathfrak{g}}_\alpha$

[$\forall\, \gamma \in \mathfrak{h}^*$, $\tilde{\mathfrak{g}}_\gamma := \{ x \in \tilde{\mathfrak{g}} : [h, x] = \gamma(h) x \ \ \forall\, h \in \mathfrak{h} \}$.]

→ If $\mathfrak{a} \subset \tilde{\mathfrak{g}}$ is an ideal, then $\mathfrak{a} = \bigoplus_{\alpha\in Q_+\setminus\{0\}} \mathfrak{a}_{\pm\alpha} \oplus (\mathfrak{a}\cap\mathfrak{h})$.

$\mathfrak{a}$ is proper $\Leftrightarrow$ $\mathfrak{a}\cap\mathfrak{h} = \{0\}$.

Let $\mathfrak{r}$ = unique max'l proper ideal. $\boxed{\mathfrak{g} := \tilde{\mathfrak{g}}/\mathfrak{r}}$ automatically simple.

→ For $i\neq j$, let $e_{ij} := \mathrm{ad}(e_i)^{1-a_{ij}} e_j \qquad f_{ij} := \mathrm{ad}(f_i)^{1-a_{ij}} f_j$.

Then $e_{ij} \in \mathfrak{r}_+ \ (\forall\, i\neq j)$ and $f_{ij} \in \mathfrak{r}_- \ (\forall\, i \neq j)$.

§1. Proposition. – Let $\mathfrak{g} \circlearrowright V$ be a f.d. repn. of $\mathfrak{g}$. Then

(i) V is $\mathfrak{h}$-diagonalizable – i.e., $V = \bigoplus_{\gamma \in \mathfrak{h}^*} V[\gamma]$ where

$$V[\gamma] := \{ v \in V : h \cdot v = \gamma(h) v \ \forall \, h \in \mathfrak{h} \}.$$

Let $P(V) \subset \mathfrak{h}^*$ be defined as: $P(V) = \{ \gamma \in \mathfrak{h}^* : V[\gamma] \neq \{0\} \}$.
(weights of V)

(ii) $P(V) \subset P := \{ \gamma \in \mathfrak{h}^* : \gamma(h_i) \in \mathbb{Z} \ \forall \, i \in I \}$ "weight lattice"

(iii) e_i and f_i act nilpotently on V.

Proof. For each $i \in I$, let $\varphi_i : sl_2 \longrightarrow \mathfrak{g}$. Then $sl_2 \overset{\pi_i}{\circlearrowright} V$

$$e \mapsto e_i$$
$$f \mapsto f_i$$
$$h \mapsto h_i$$

and hence (see Lecture 26 - page 1) h_i acts diagonalizably on V and
(from sl_2- repn th.)

$$\gamma \in P(V) \Rightarrow \gamma(h_i) \in \mathbb{Z}. \quad \underline{\text{(ii) follows.}}$$

As $[h_i, h_j] = 0$, $\{h_i\}_{i \in I} \subset \mathrm{End}_{\mathbb{C}}(V)$ can be simultaneously diagonalized. This proves (i).

For (iii), note that $e_i : V[\gamma] \to V[\gamma + \alpha_i]$ and $|P(V)| < \infty$
$(\dim V < \infty)$

$\Rightarrow$ e_i acts nilpotently (similarly for f_i's) $\square$

§2. Integrable repns. and Weyl group symmetry.

Defn: A $\mathfrak{g}$-repn., V is said to be *integrable* if $\forall\, i \in I$,

e_i and f_i act *locally nilpotently* on V.

($\forall\, v \in V$, $\exists\, N \gg 0$ (depending on i & v) s.t.
$e_i^N \cdot v = 0$, $f_i^N \cdot v = 0$.)

Theorem. – Let $\mathfrak{g} \circlearrowright V$ be an $\mathfrak{h}$-diagonalizable repn which is integrable. Then, $\forall\, \gamma \in P(V)$ and $w \in W$ (Weyl group)

$$V[\gamma] \cong V[w(\gamma)]$$

(recall $W = \langle s_i \rangle_{i \in I}$ and $s_i \circlearrowright \mathfrak{h}^*$ by $s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$ $\forall j \in I$.)

Proof. Let $i \in I$ be fixed. Define (see Thm §2 - Lecture 26).

$$\sigma_{i;V} = \exp(e_i)\exp(-f_i)\exp(e_i) : V \to V \quad \in GL(V).$$

(makes sense by local nilpotence).

Claim. – $\exp(\mathrm{ad}(e_i))\exp(-\mathrm{ad}(f_i))\exp(\mathrm{ad}(e_i)) : h \mapsto s_i(h) = h - \alpha_i(h) h_i$ $\forall\, h \in \mathfrak{h}$.

(Pf. of the claim – see Pages 3,4 of Lecture 26).

Hence, if $v \in V[\gamma]$ ($w = s_i$), $u = \sigma_i(v)$, then

$$\sigma_i(h \cdot v) = \gamma(h)\, \sigma_i(v) = \gamma(h) \cdot u$$

$$\| $$

$$(\sigma_i h \sigma_i^{-1})(u) = s_i(h)(u)$$

$\Rightarrow h' \cdot u = \gamma(s_i(h'))\, u = (s_i \cdot \gamma)(h')\, u$

$\Rightarrow u \in V[s_i \gamma]$ □

§3. Finite-dimensionality of $\mathfrak{g}$.

- Recall $R := \bigcup_{i\in I} W\cdot\alpha_i \subset \mathfrak{h}^* \setminus \{0\}$. (see Lecture 27 - §3,4)

$|R| < \infty$ and $R = R_+ \sqcup R_-$ $\quad (R_+ = R \cap Q_+)$

$|W| < \infty$ $\quad (R_- = R \cap (-Q_+) = -R_+)$

- By Serre rel^{n}s -

<u>Lemma.</u> - $\mathfrak{g} \circlearrowright \mathfrak{g}$ via adjoint map, is $\mathfrak{h}$-diagonalizable and integrable.

<u>Proof.</u> - $\mathfrak{h}$-diagonalizability follows from Thm §2 Lecture 29.

$ad(e_i)$ acts locally nilpotently on gen's $\{h_j, e_j, f_j\}_{j\in I}$ — by Serre rel^{n}s

and
$$\frac{ad(e_i)^N}{N!}\cdot[X,Y] = \sum_{a=0}^{N}\left[\frac{ad(e_i)^a}{a!}\cdot X\,,\,\frac{ad(e_i)^{N-a}}{(N-a)!}\cdot Y\right]$$
$\Rightarrow$ $ad(e_i)$ acts locally nilpotently on $\mathfrak{g}$. $\square$

<u>Cor.</u> - (i) Let $P(\mathfrak{g}) = \{\alpha\in\mathfrak{h}^* : \mathfrak{g}_\alpha \neq \{0\}\}$. Then $P(\mathfrak{g})$ is W-invariant.

$$P(\mathfrak{g}) = \{0\} \sqcup P(\mathfrak{g})_+ \sqcup P(\mathfrak{g})_-$$

(ii) $P(\mathfrak{g})_\pm = R_\pm$.

<u>Proof.</u> (i) follows from the triangular dec. of $\tilde{\mathfrak{g}}$ and $\mathfrak{r}\subset\tilde{\mathfrak{g}}$

$$\mathfrak{g} = \tilde{\mathfrak{g}}/\mathfrak{r} = \left(\tilde{\mathfrak{n}}_-/\mathfrak{r}_-\right)\oplus\mathfrak{h}\oplus\left(\tilde{\mathfrak{n}}_+/\mathfrak{r}_+\right).$$

W-invariant by Lemma above and Thm §2 on previous page.

(ii) As $\mathfrak{g}_{\alpha_i} = \mathbb{C} \cdot e_i \ (\forall i \in I)$ we get

$$\mathfrak{g}_{w(\alpha_i)} \cong \mathfrak{g}_{\alpha_i} \quad (\forall w \in W) \quad \Rightarrow \quad \dim \mathfrak{g}_\alpha = 1 \quad \forall \alpha \in R.$$

So, $R_\pm \subset P(\mathfrak{g})_\pm$. Assume $R_+ \subsetneq P(\mathfrak{g})_+$ and let

$\beta \in P(\mathfrak{g})_+ \setminus R_+$ be of smallest <u>height</u> $\left(ht\left(\sum_{i \in I} k_i \alpha_i\right) := \sum_{i \in I} k_i \right)$

<u>Note</u>: $\beta \neq \alpha_i$ and hence $s_i(\beta) = \beta - \beta(h_i)\alpha_i \in P(\mathfrak{g})_+ \setminus R_+$ (by W-invariance)

By height minimality, $\beta(h_i) \leq 0 \quad \forall \ i \in I$. So, if $\beta = \sum_{i \in I} k_i \alpha_i \in Q_+$

then, $(\beta, \beta) = \sum_{i \in I} (\beta, \alpha_i) \, k_i = \sum_{i \in I} k_i d_i \beta(h_i) \leq 0$

Contradicting positive-definite-ness of $(\cdot, \cdot)$ on E^*. $\square$

§4. <u>Summary</u>. -

$$\mathfrak{g} = \left(\bigoplus_{\alpha \in R_-} \mathfrak{g}_\alpha \right) \oplus \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in R_+} \mathfrak{g}_\alpha \right)$$

$$\dim \mathfrak{h} = |I| \quad ; \quad \dim \mathfrak{g}_\alpha = 1 \quad \forall \alpha \in R.$$

So
$$\boxed{\begin{aligned} \dim(\mathfrak{g}) &= |I| + |R| \\ &= |I| + 2|R_+| \end{aligned}}$$