Lecture 31.

Recall:

A : Cartan matrix $= (a_{ij})_{i,j\in I}$

$D = \mathrm{diag}(d_i : i \in I)$. symmetrizing integers.

Root system

$\{h_i\}_{i\in I} \subset E$ basis $/\mathbb{R}$

$\{\alpha_j\}_{j\in I} \subset E^* : \alpha_j(h_i) = a_{ij}$

$(\cdot,\cdot)$ on E^* : $(\alpha_i, \alpha_j) = d_i a_{ij}$

$(\cdot,\cdot)$ on E : $(h_i, h_j) = a_{ij} d_j^{-1}$

$\phi : E^* \to E$ isometry

$\phi(\alpha_i) = d_i h_i$

Reflections: $s_i(\gamma) = \gamma - \gamma(h_i)\alpha_i \quad \forall \gamma \in \mathfrak{h}^*$

$s_i(h) = h - \alpha_i(h) h_i \quad \forall h \in \mathfrak{h}.$

$W = \langle s_i \rangle \circlearrowright E$ and E^* ; preserving

$(\cdot,\cdot)$ and hence ϕ is W-equivariant.

$$R = \bigcup_{i\in I} W\alpha_i \subset E^* \setminus \{0\}.$$

Lie alg $\mathfrak{g} = \tilde{\mathfrak{g}}/\mathfrak{r}$ Finite-dim'l simple LA $/\mathbb{C}$

$\mathfrak{h} := E \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}$-span of $\{h_i\}_{i\in I}$.

$\alpha_j \in \mathfrak{h}^*$. $\quad Q_+ := \sum_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i \subset \mathfrak{h}^*$.

$$\mathfrak{g} = \left(\bigoplus_{\alpha\in R_+} \mathfrak{g}_{-\alpha}\right) \oplus \mathfrak{h} \oplus \left(\bigoplus_{\alpha\in R_+} \mathfrak{g}_\alpha\right)$$

Generators : $\mathfrak{h}$, $\{e_i, f_i\}_{i\in I}$.

Rel^{n}s (we don't know if these are all yet):

$\mathfrak{h} \subset \mathfrak{g}$ is abelian Lie subalg.

$[h, e_i] = \alpha_i(h) e_i$

$[h, f_i] = -\alpha_i(h) f_i$ $\quad$ (i.e. $e_i \in \mathfrak{g}_{\alpha_i}$, $f_i \in \mathfrak{g}_{-\alpha_i}$)

$[e_i, f_j] = \delta_{ij} h_i$

$\forall i \neq j \quad \mathrm{ad}(e_i)^{1-a_{ij}} . e_j = 0$

$\mathrm{ad}(f_i)^{1-a_{ij}} . f_j = 0$ $\quad$] Serre rel^{n}s

$R = R_+ \sqcup R_-$ where $R_+ = (R \cap Q_+)$.

$R_- = -R_+$.

Important result. If $\mathfrak{g} \circlearrowright V$ is an $\mathfrak{h}$-diagonalizable repn. where

$\{e_i, f_i\}_{i\in I}$ act locally nilpotently, then

$$V[\gamma] \cong V[w\gamma] \quad \forall \gamma \in P(V), \; w \in W.$$

§1. Our next goal is to study finite-dim'l, irred. repns. of $\mathfrak{g}$.

<u>Lemma.</u> – Every f.d. irred $\mathfrak{g}$-repn. is a <u>highest-weight repn</u> – i.e.,

$\mathfrak{g} \circlearrowright V$ f.d. irreducible $\Rightarrow$ (i) V is $\mathfrak{h}$-diagonalizable (see Prop §1 of Lecture 30)

(ii) $\exists ! \, \lambda \in P_+$ s.t. $\dim V[\lambda] = 1$.

[uniqueness will be proved next.]

- $P(V) \subseteq \lambda - Q_+$.
  So $e_i \cdot v = 0 \quad \forall \, v \in V[\lambda], \, i \in I$.
- $V = \mathrm{Span}\left\{ f_{i_1} \cdots f_{i_n} v \;\middle|\; \begin{array}{l} n \in \mathbb{Z}_{\geq 0} \\ i_1, \dots, i_n \in I \end{array} \right\}$.
  $(0 \neq v \in V[\lambda])$

<u>Proof.</u> – (i) was proved, for any f.d. repn – irred. or not – in Prop §1 of last lecture. Let $V = \bigoplus_{\mu \in P(V)} V[\mu]$ be the weight space dec.

$P(V) \subset P = \{\gamma \in \mathfrak{h}^* : \gamma(h_i) \in \mathbb{Z} \;\; \forall i \in I\}$

(Prop §1 L.30)

$\dim V < \infty \;\Rightarrow\; |P(V)| < \infty$.

$\Rightarrow \exists \, \lambda \in P(V)$ s.t. $\lambda + \alpha_i \notin P(V) \;(\forall i \in I)$.

Thus $e_i \cdot v = 0 \quad \forall \, i \in I$ and $v \in V[\lambda]$. Let $0 \neq v \in V[\lambda]$.

Define $U := \mathrm{Span}\left\{ f_{i_1} \cdots f_{i_n} \cdot v \;\middle|\; \begin{array}{l} n \in \mathbb{Z}_{\geq 0} \\ i_1, \dots, i_n \in I \end{array} \right\} \subset V$.

<u>Claim:</u> U is a subrepn. Hence, by irred. of V, $U = V$ and (ii) follows.

[$v \in U$, so $U \neq \{0\}$).

Pf. U is clearly stable under $\{f_i\}_{i \in I}$ "lowering operators".

Also, $h \cdot (f_{i_1} \cdots f_{i_n} \cdot v) = \left(\lambda - \sum_{j=1}^{n} \alpha_{i_j}\right)(h) \left(f_{i_1} \cdots f_{i_n} v\right)$

Finally:

$$e_i \cdot (f_{i_1} \cdots f_{i_n} v) = (f_{i_1} \cdots f_{i_n}) \cdot \cancel{e_i v}^{\,0} + \underbrace{\sum_{j=1}^{n} f_{i_1} \cdots f_{i_{j-1}} [e_i, f_{i_j}] f_{i_{j+1}} \cdots f_{i_n} \cdot v}_{\text{in } U}$$

Hence, U is a subrepn. $\square$

§2. Definition. – Verma Module. – Let $\lambda \in \mathfrak{h}^*$ and define a $\mathfrak{g}$-repn., denoted by M_λ, as the universal highest weight repn., of highest weight λ. That is, (i) $M_\lambda[\lambda] = \mathbb{C} \cdot \mathbb{1}_\lambda$ (highest wt. vector)

$$P(M_\lambda) \subset \lambda - Q_+.$$

(ii) If $\mathfrak{g} \circlearrowright V$ is any repn. and $v \in V[\lambda]$ is s.t. $e_i v = 0 \ \forall i \in I$, $h \cdot v = \lambda(h) \cdot v \ \forall h \in \mathfrak{h}$,

then $\exists!$ $\mathfrak{g}$-intertwiner $\varphi : M_\lambda \longrightarrow V$ s.t. $\varphi(\mathbb{1}_\lambda) = v$.

Note – (ii) $\Rightarrow$ If $\mathfrak{g} \circlearrowright V$ is a h.w. repn. of h.w. λ, then $M_\lambda \twoheadrightarrow V$ (since $\dim V[\lambda] = 1$ & V is generated by $V[\lambda]$).

This proves uniqueness of Verma Modules.

Existence: $M_\lambda := \mathcal{U}(\mathfrak{g}) \Big/$ left ideal gen. by e_i $(i \in I)$, $h - \lambda(h) \cdot 1$ $(h \in \mathfrak{h})$

$\mathbb{1}_\lambda :=$ coset of $1 \in \mathcal{U}(\mathfrak{g})$.

Note. – By PBW theorem; $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$

$$\Rightarrow \mathcal{U}(\mathfrak{g}) \cong \mathcal{U}(\mathfrak{n}_-) \otimes \mathcal{U}(\mathfrak{h}) \otimes \mathcal{U}(\mathfrak{n}_+) \quad [\text{vector space iso.}]$$

$$\Rightarrow \quad M_\lambda \cong \mathcal{U}(\mathfrak{n}_-) \quad \text{as vector spaces}$$

$$f_{i_1}\cdots f_{i_n}\cdot \mathbb{1}_\lambda \longleftrightarrow f_{i_1}\cdots f_{i_n} \qquad \psi : \mathcal{U}(\mathfrak{n}_-) \to M_\lambda .$$

and : $\psi : \mathcal{U}(\mathfrak{n}_-)_{-\beta} \cong M_\lambda[\lambda-\beta] \quad \forall\ \beta \in Q_+ .$ $\square$

§3. Properties of M_λ. – (i) $M_\lambda[\lambda] = \mathbb{C}\cdot \mathbb{1}_\lambda$.

$$M_\lambda = \bigoplus_{\beta\in Q_+} M_\lambda[\lambda-\beta]$$

(ii) A subrepn. $V \subset M_\lambda$ is proper $\Longleftrightarrow$ $V \cap M_\lambda[\lambda] = \{0\}$.

(i.e. $V \subset \bigoplus_{\beta\in Q_+\setminus\{0\}} M_\lambda[\lambda-\beta]$)

(iii) $\dim M_\lambda[\lambda-\beta] = \#\left\{ a_1,\dots,a_N \in \mathbb{Z}_{\geq 0}^N : \sum_{j=1}^N a_j \beta_j = \beta \right\} < \infty.$

Let $R_+ = \{\beta_1, \dots, \beta_N\}$.

(called Kostant's partition fn. $P_K(\beta) = \#$ of ways of writing β as a sum of positive roots.)

Proof. – (i) and (ii) are obvious. For (iii) note that

$$\mathfrak{n}_- = \bigoplus_{\alpha\in R_+} \underbrace{\mathfrak{g}_{-\alpha}}_{\text{1-dim'l}} \Rightarrow \mathfrak{n}_- \text{ has a basis labelled by } R_+$$

$$\{ f_{\beta_j} \in \mathfrak{g}_{-\beta_j} : 1 \leq j \leq N \}$$

$\beta_1, \dots, \beta_N$: an arbitrary enumeration of R_+.

By PBW Thm. $\left\{ f_{\beta_1}^{a_1} \cdots f_{\beta_N}^{a_N} : \underline{a} \in \mathbb{Z}_{\geq 0}^N \right\}$ is a basis of $\mathcal{U}(\mathfrak{n}_-)$.

From the iso $\mathcal{U}(\mathfrak{n}_-) \to M_\lambda$ above,

$$f_{i_1} \cdots f_{i_n} \mapsto f_{i_1} \cdots f_{i_n} \mathbb{1}_\lambda \in M_\lambda [\lambda - \sum_{j=1}^{n} \alpha_{i_j}]$$

we get :

$$\dim M_\lambda [\lambda - \beta] = \dim \mathcal{U}(\mathfrak{n}_-)_{-\beta}$$

$$= \# \{ \underline{a} \in \mathbb{Z}_{\geq 0}^N : \sum_{j=1}^{N} a_j \beta_j = \beta \} .$$

$\square$

§4. By §3 (ii) above, $\exists !$ max'l proper $\mathfrak{g}$-subrepn. $\mathcal{J}_\lambda \subset M_\lambda$.

Define $L_\lambda := M_\lambda / \mathcal{J}_\lambda$ irreducible, highest weight repn of h.w λ.

<u>Uniqueness</u>: If V is any irred; h.w. repn. of $\mathfrak{g}$, with h.w. λ,

then $\pi : M_\lambda \twoheadrightarrow V$ by universal prop. of M_λ.

$\mathbb{1}_\lambda \longmapsto v$

$(V[\lambda] = \mathbb{C} \cdot v)$

[i.e. $V[\lambda]$ is 1-dim'l.
$e_i = 0$ on $V[\lambda]$ $\forall i$
V is gen. by $V[\lambda]$
& <u>V is irred.</u>]

$\operatorname{Ker}(\pi) \subset \mathcal{J}_\lambda \Rightarrow \pi : L_\lambda \to V$.
factors through.

$\pi \neq 0$ (it is surj.), both L_λ & V irred $\Rightarrow L_\lambda \cong V$. This establishes uniqueness (see Lemma §1 -page2 above).

§5. <u>Theorem.</u> $\operatorname{Irred}_{fd}(\mathfrak{g}) = \{ L_\lambda : \lambda \in P_+ \}$

i.e. L_λ constructed above is f.d. $\iff \lambda \in P_+$

(recall: $P_+ = \{ \gamma \in \mathfrak{h}^* : \gamma(h_i) \in \mathbb{Z}_{\geq 0} \ \forall i \in \underline{I} \}$)
"dominant weights"

Moreover, the following rel^{n}s hold in L_λ.

$$f_i^{\lambda(h_i)+1} \cdot \mathbb{1}_\lambda = 0 \qquad \forall\, i \in I .$$

<u>Proof.</u> - If $\dim L_\lambda < \infty$ then $\forall\, i$, $\underset{\langle e_i, f_i, h_i\rangle}{\overset{\shortparallel}{sl_2^{(i)}}} \circlearrowright L_\lambda$ implies $\begin{array}{l} e_i \cdot \mathbb{1}_\lambda = 0 \\ h_i \cdot \mathbb{1}_\lambda = \lambda(h_i)\, \mathbb{1}_\lambda \end{array}$

$\lambda(h_i) \in \mathbb{Z}_{\geq 0}$ and $f_i^{\lambda(h_i)+1} \cdot \mathbb{1}_\lambda = 0$.

Conversely, if $\lambda \in P_+$, then $\forall\, i \in I$, $f_i^{\lambda(h_i)+1} \cdot \mathbb{1}_\lambda \in M_\lambda[\lambda - (\lambda(h_i)+1)\alpha_i]$

is a highest weight vector (sl_2 - rep. th.) : this is because

$$e_j \cdot f_i^{\lambda(h_i)+1} \cdot \mathbb{1}_\lambda = \underbrace{f_i^{\lambda(h_i)+1} \cdot e_j \cdot \mathbb{1}_\lambda}_{=0} + [e_j, f_i^{\lambda(h_i)+1}] \cdot \mathbb{1}_\lambda$$

$$= \delta_{ij}\, f_i^{\lambda(h_i)} \cdot (h_i - \lambda(h_i)) \cdot \mathbb{1}_\lambda = 0 \qquad \forall\, j \in I.$$

$\Rightarrow$ J_λ = $\mathfrak{g}$ - subrepn of M_λ gen. by $\{f_i^{\lambda(h_i)+1} \cdot \mathbb{1}_\lambda : i \in I\}$

is proper, hence $J_\lambda \subset \mathcal{J}_\lambda$ (the unique, proper, max'l subrepn.).

$$\Rightarrow \widetilde{L}_\lambda := M_\lambda / J_\lambda \twoheadrightarrow L_\lambda = M_\lambda / \mathcal{J}_\lambda .$$

<u>Claim.</u> - $\dim \widetilde{L}_\lambda < \infty$. (hence, $\dim L_\lambda < \infty$).

<u>Pf.</u> - $\widetilde{L}_\lambda$ is a h.w. repn. of h.w. λ (being quotient of one) by

e_i's act locally nilpotently on M_λ (hence on $\widetilde{L}_\lambda$) by

highest weight condition.

- f_i acts locally nilpotently on $\widetilde{L}_\lambda$. This is because $f_i^{\lambda(h_i)+1} \cdot \mathbb{1}_\lambda = 0$ in $\widetilde{L}_\lambda$; ~~and~~ $\widetilde{L}_\lambda$ is spanned by $\left\{ f_{i_1} \cdots f_{i_n} \cdot \mathbb{1}_\lambda \,\middle|\, \begin{matrix} n \in \mathbb{Z}_{\geq 0} \\ i_1, \dots, i_n \in I \end{matrix} \right\}$

and $\mathrm{ad}(f_i)$ is nilpotent on f_j's (Serre rel^{n}s).

$$\frac{f_i^N}{N!} \left(f_{i_1} \cdots f_{i_n} \cdot \mathbb{1}_\lambda \right) = \sum_{\substack{a_1, \dots, a_{n+1} \in \mathbb{Z}_{\geq 0} \\ a_1 + \dots + a_{n+1} = N}} \left(\frac{(\mathrm{ad}\, f_i)^{a_1}}{a_1!} \cdot f_{i_1} \right) \cdots \left(\frac{(\mathrm{ad}\, f_i)^{a_n}}{a_n!} \cdot f_{i_n} \right) \cdot \left(\frac{f_i^{a_{n+1}}}{a_{n+1}!} \cdot \mathbb{1}_\lambda \right)$$

$$= 0 \quad \text{for } N \gg 0.$$

So, $P(\widetilde{L}) \subset \lambda - Q_+$ is W-invariant (Thm. §2. Lect 30).

<u>Recall</u>: $\forall\ a \in E$ $(\mathfrak{h} = E \otimes_{\mathbb{R}} \mathbb{C})$, $\exists\, w \in W$ s.t. $\underbrace{w(a)}_{\text{unique.}} \in \overline{C_0}$

(i.e. $\alpha_i(w(a)) \geq 0$ $\forall i \in I$).

$\Rightarrow \forall\ \beta \in P(\widetilde{L})$, $\exists\ \beta_0 \in W \cdot \beta$ s.t. $\beta_0(h_i) \geq 0$ $\forall\ i \in I$.

Hence, $P(\widetilde{L}) \subset W \cdot \underbrace{(\lambda - Q_+) \cap P_+}_{\text{finite set}}$ $\Rightarrow \dim \widetilde{L} < \infty$.

$\square$